EE1 Oscillations
Course Outline

- Lecture 1: Simple harmonic motion
- Lecture 2: The simple pendulum
- Lecture 3: Damped oscillations
- Lecture 4: Electrical oscillations

Intro

Any motion that repeats itself at regular intervals is said to be ***periodic***

Examples

- Vibrations on a guitar string
- Speaker cone
- Swinging of a pendulum
- Vibrations of atoms in solids

Oscillations

An oscillation is a periodic fluctuation in the value of a physical quantity above and below some central or equilibrium value

Mechanical Oscillations

Examples listed on previous slide.

Body undergoes linear or angular displacement

Non-mechanical Oscillations

Involve variation of quantities such as voltage in electrical circuits or electric and magnetic fields in radio and TV signals

Simple Harmonic Motion (SHM)

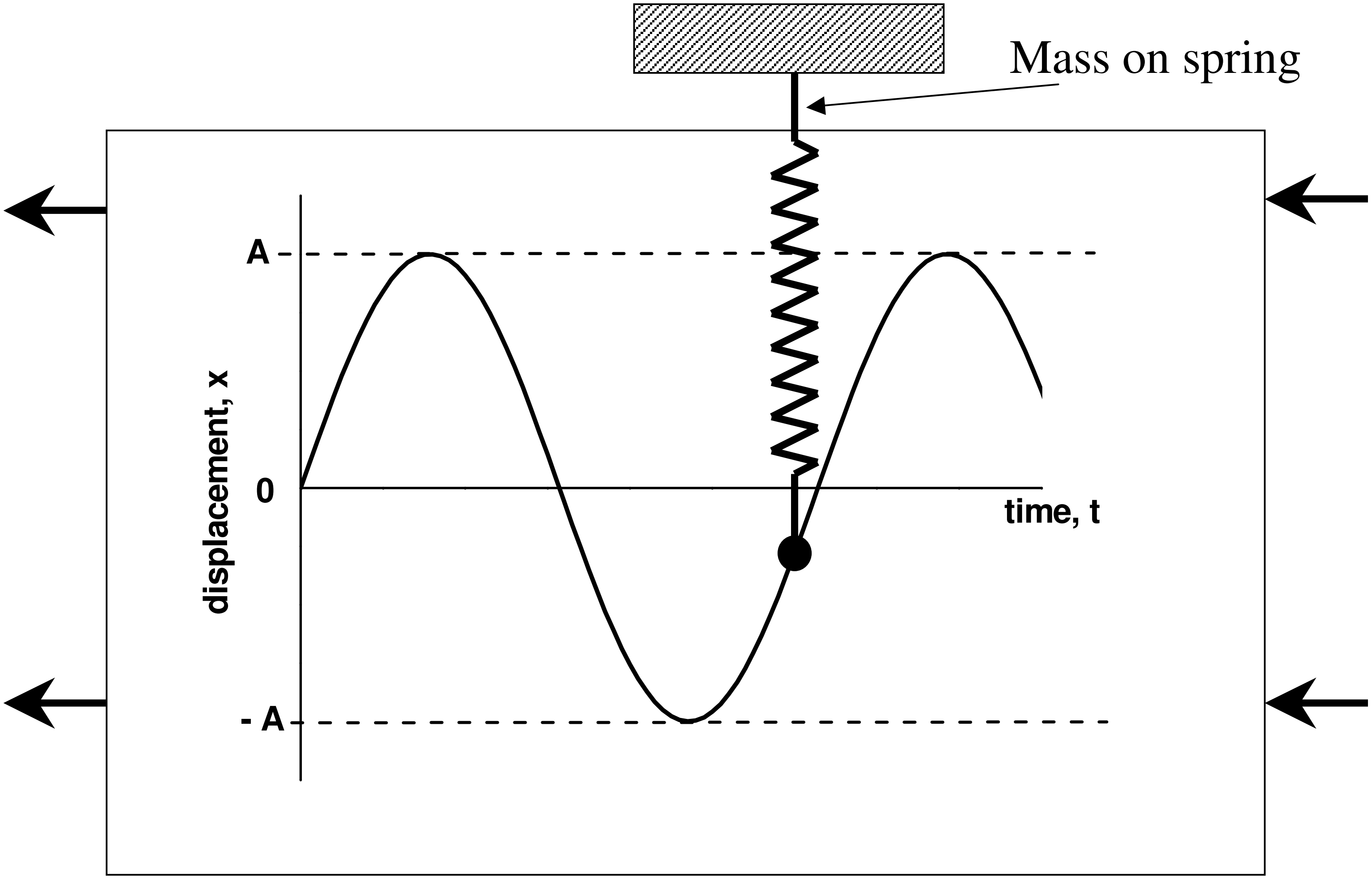
Idealised: First consider the case of oscillations, which occur without loss of energy

Also the physical system must have a position of stable equilibrium

And the restoring force must be proportional to the displacement

Example of SHM

Consider a mechanical system of a block (mass = m) attached to a spring.
To study how the displacement *x* of a block varies with time, we can record the motion on a strip of paper that moves with constant speed



Strip of paper moving with constant speed

In the absence of friction, the block oscillates between the extreme values +A and –A.

A is the *amplitude* of the oscillations.

The displacement from equilibrium is given by:

$$\mathbf{x(t) = A\sin\omega t} \qquad \mathbf{(1)}$$

ω is measured in radians/sec and is called the *angular frequency*

One cycle corresponds to 2π radians and is completed in one *period* T (seconds)

$$\omega T = 2\pi$$

$$\mathbf{\omega = 2\pi/T = 2\pi f} \qquad \mathbf{(2)}$$

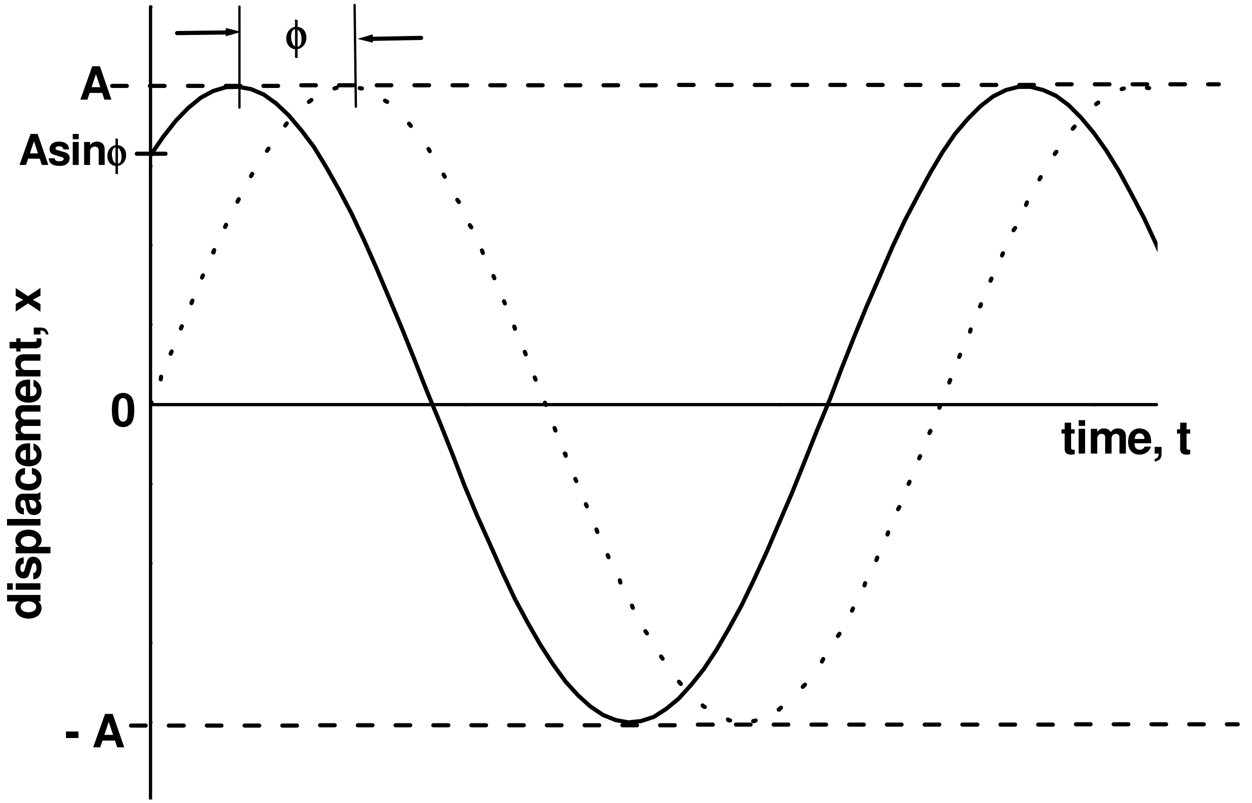
f = 1/T is the frequency of the oscillations

(sec^{-1} or hertz, Hz)

In the previous figure, the block was at x = 0 at t = 0
In general this will not be the case, so we write:

$$x(t) = A\sin(\omega t + \phi) \qquad (3)$$

The argument ($\omega t + \phi$) is called the PHASE



φ is the **PHASE CONSTANT** or the **INITIAL PHASE** and is measured in **radians.** Its value is determined by the initial conditions in the way motion was started

Simple harmonic oscillator

Any system in which the variation in time of a physical quantity x is given by equation 3 is called a SIMPLE HARMONIC OSCILLATOR (SHO)

A SHO has the following characteristics:

1. **Amplitude A is constant**
2. **f and T are independent of amplitude**
3. **Time dependence of x is a sinusoidal (harmonic) function of frequency (ω).**

Look at equation 3 again:

So:

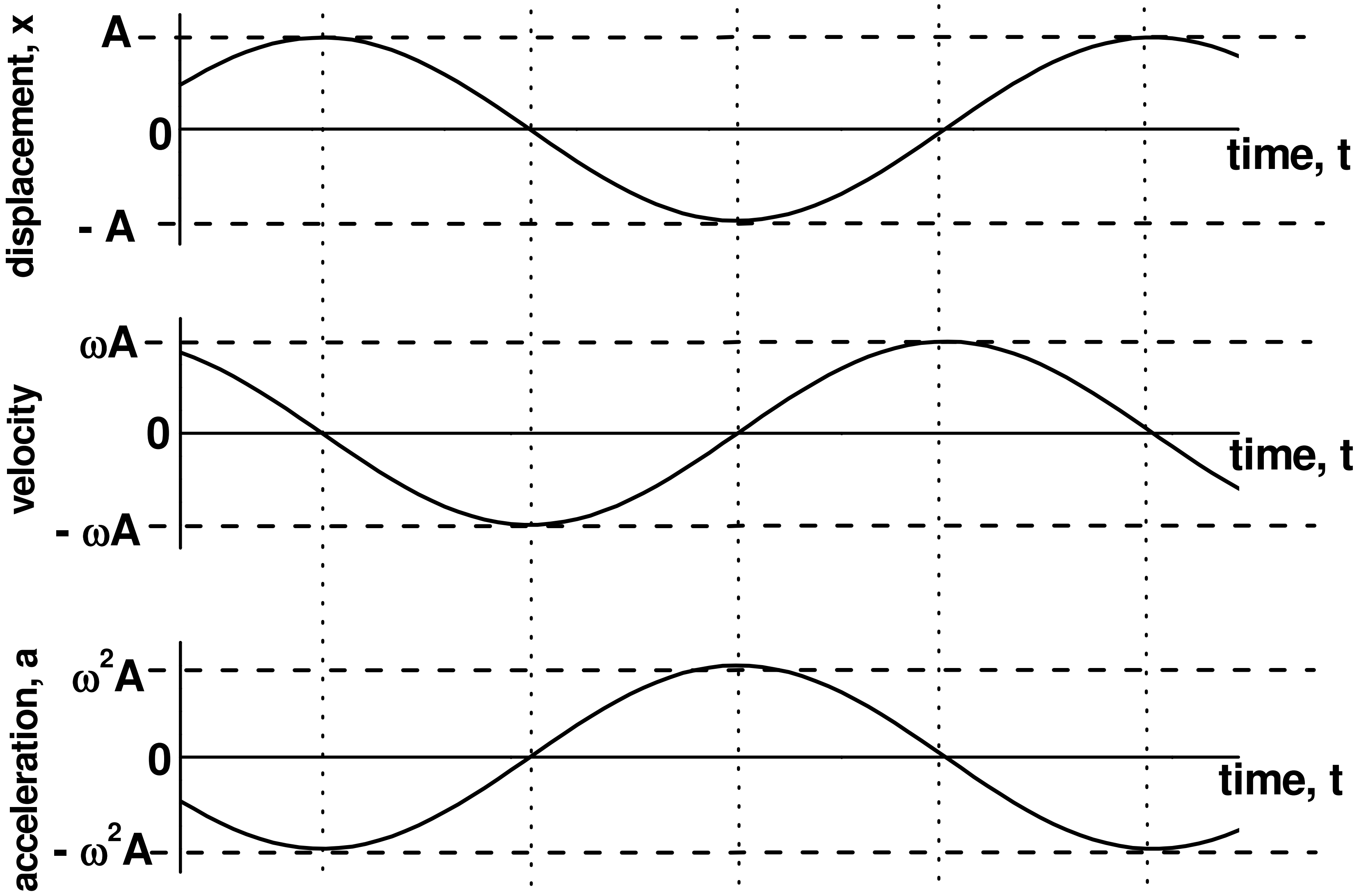
displacement $\mathbf{x(t) = A \sin(\omega t + \phi)}$

velocity $$\mathbf{dx/dt = \omega A \cos(\omega t + \phi)} \tag{4}$$

acceleration $$\mathbf{d^2x/dt^2 = - \omega^2 A \sin(\omega t + \phi)} \tag{5}$$

$$\mathbf{a = - \omega^2 x} \tag{6}$$

Acceleration is proportional to displacement and opposite in direction



The figure above shows:

- Extreme values of **velocity (v) are ±ωA**, which occur when ***x*** **= 0**
- Extreme values of **acceleration (a) are ±ω^2A**, occurs when ***x*** **= ±A**

From equations 5 and 6

$$\mathbf{d^2}x\mathbf{/dt^2 + \omega^2}x\mathbf{ = 0} \qquad \mathbf{(7)}$$

Equation 7 characterises all types of SHO or SHM whether mechanical or non-mechanical.

Equation 3 is a solution of this differential equation

A and ϕ are determined by the initial conditions

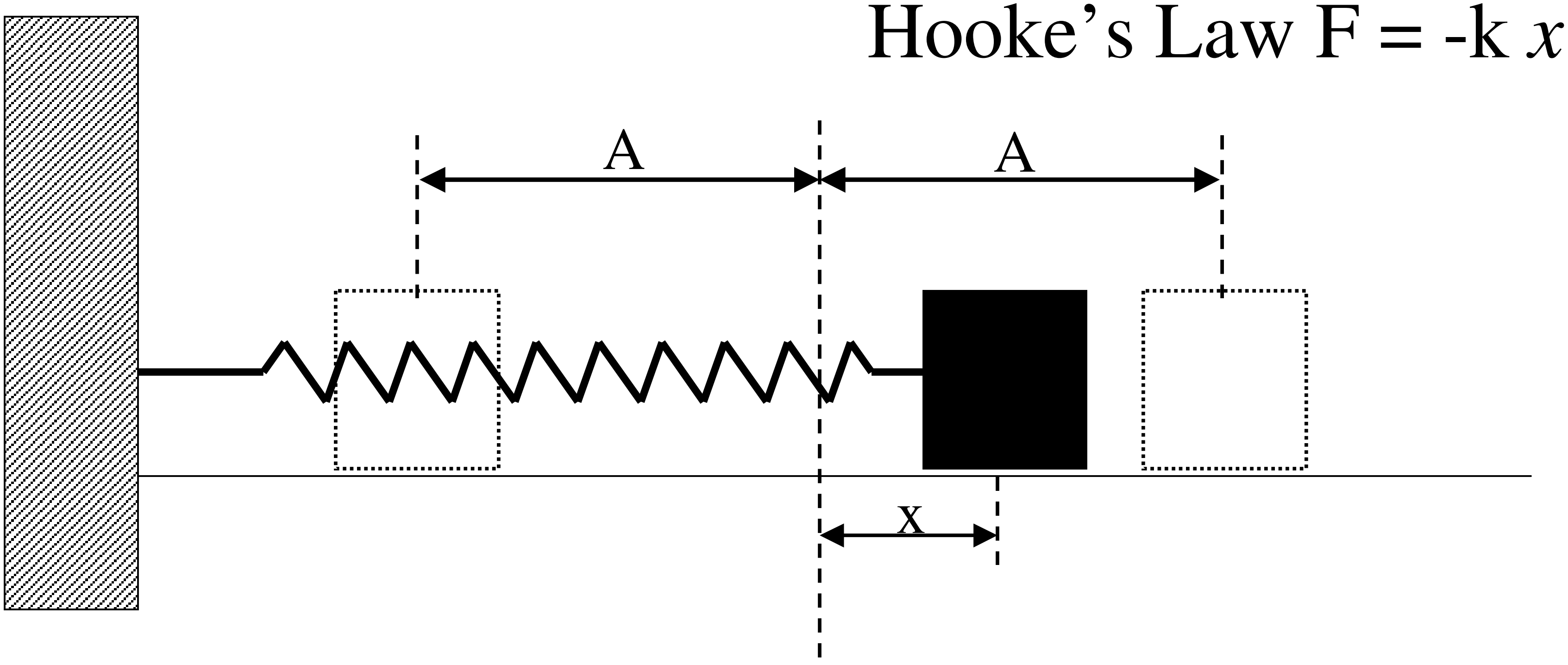
The block- spring system

Consider a block of mass m oscillating at the end of a massless spring. No friction forces are present.

At position x, the net force on the block is exerted by the spring and it is given by Hooke's Law:

$$F = -k\,x$$

Hooke's Law F = -k *x*



x = 0

When *x* is positive, F is directed towards *x* = 0 and is negative

When *x* is negative, F is directed towards *x* = 0 and is positive

F has the tendency to bring mass to a stable positions (x = 0)
RESTORING FORCE
k is called the SPRING CONSTANT and is a measure of the stiffness of the spring

Force F causes an acceleration of the block.

Newton's 2^{nd} Law ($F = ma$) gives:

$$F = ma = -kx$$

$$\mathbf{a = -(k/m)\ x} \qquad \mathbf{(8)}$$

The acceleration is proportional to the displacement and is directly opposite to it

Since $a = d^2x/dt$ we have:

$$\mathbf{d^2x/dt + (k/m)\ x = 0} \qquad \mathbf{(9)}$$

Comparing equation 7 with equation 9, we see that the block executes SHM with

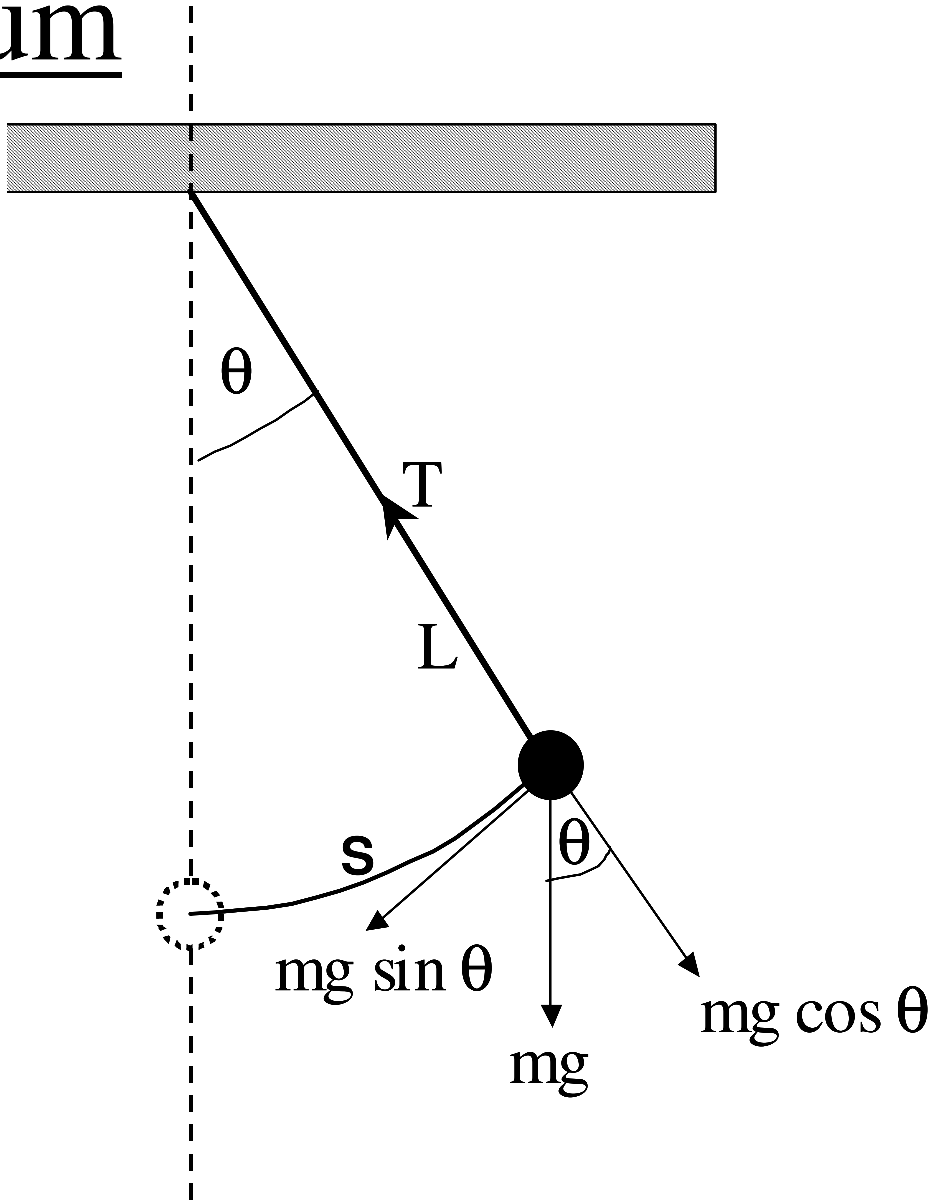
angular frequency $\boldsymbol{\omega = \sqrt{(k/m)}}$

and period $\mathbf{T = 2\pi/\omega = 2\pi \sqrt{(m/k)}} \qquad \mathbf{(10)}$

Period is independent of A

Period is increasing with mass of block and decreases for stiffer spring (larger k).

The simple pendulum



Idealised system.
Point mass m is suspended from a fixed support with a massless string of length L. At position of deflection θ, distance along the arc from the lowest point (stable equilibrium point) is s.

$$s = L\,\theta \qquad (11)$$

θ is in radians.

The simple pendulum (cont.)

The net force on the mass along the tangent is mg sin θ

Newton's 2nd Law gives

$$- mg \sin\theta = m\, d^2s/dt^2 = m\, L\, d^2\theta/dt^2$$

$$\mathbf{d^2\theta/dt^2 = -g/L \sin\theta} \qquad (12)$$

For small values of θ, $\sin\theta \approx \theta$

$$\mathbf{d^2\theta/dt^2 + (g/L)\,\theta = 0} \qquad (13)$$

Equation 13 represents the SHM with angular frequency

$$\mathbf{\omega^2 = g/L} \qquad \textbf{so} \qquad \mathbf{\omega = \sqrt{(g/L)}}$$

$$\mathbf{T = 2\pi/\omega = 2\pi\sqrt{(L/g)}} \qquad \mathbf{(14)}$$

The period does not depend on either mass or amplitude of oscillations.
Equation 14 is only true for small values of $\theta \approx < 20^o$ (1/3 radian).
For larger angles, the solution (motion) is **not** SHM and period depends on amplitude

Energy in SHM

In SHM: restoring force α displacement

e.g. spring mass system

$$F = -kx \quad ; \qquad x(t) = A\sin(\omega t + \phi)$$

Potential energy

$$U = - \ F.dx = \tfrac{1}{2} kx^2$$

$$\mathbf{U = \tfrac{1}{2} kA^2 \sin^2(\omega t + \phi)}$$

Kinetic energy

$$K = \tfrac{1}{2} mv^2 = \tfrac{1}{2} m(dx/dt)^2$$

$$\mathbf{K = \tfrac{1}{2} mA^2 \omega^2 \cos^2(\omega t + \phi)}$$

But

$$\omega^2 = k/m \quad \text{or} \quad m\omega^2 = k$$

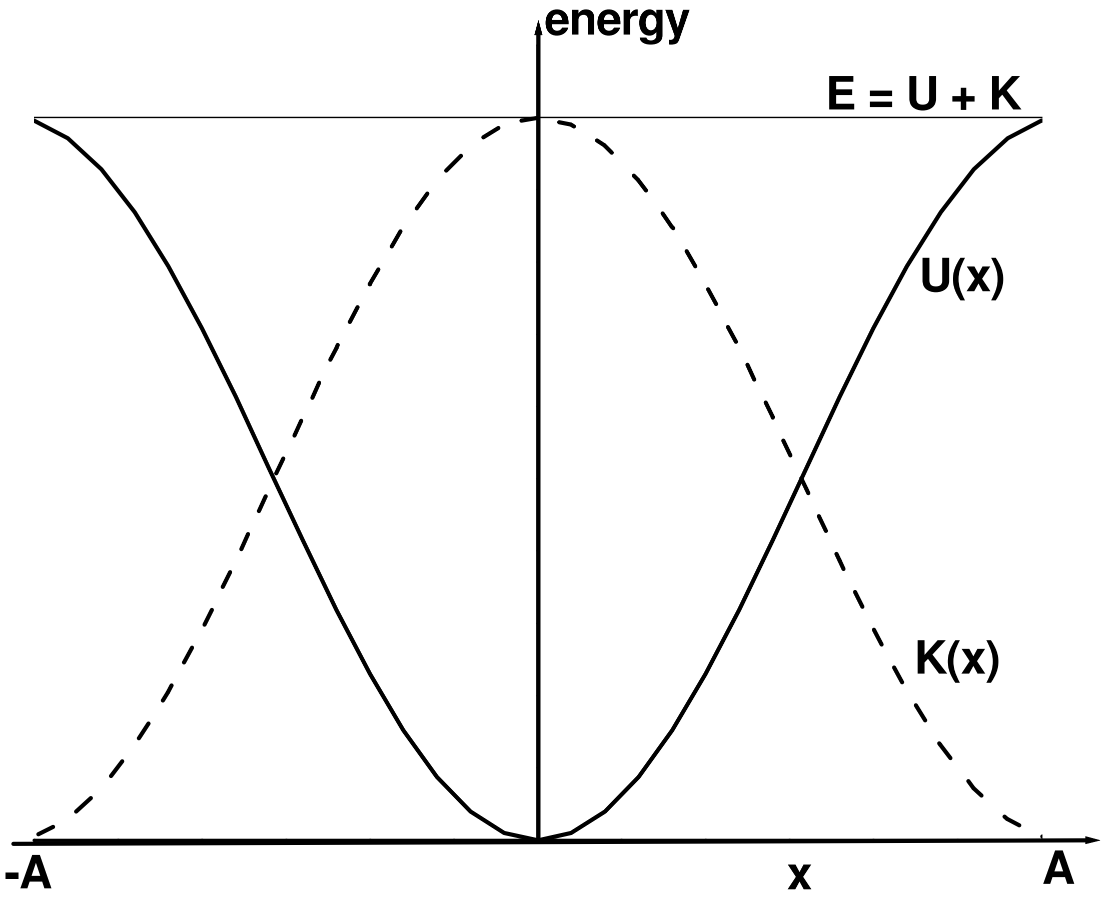
$$\mathbf{K = \tfrac{1}{2} A^2 k \cos^2(\omega t + \phi)}$$

Total energy

$$\mathbf{E = U + K = \tfrac{1}{2} kA^2 (\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi))}$$

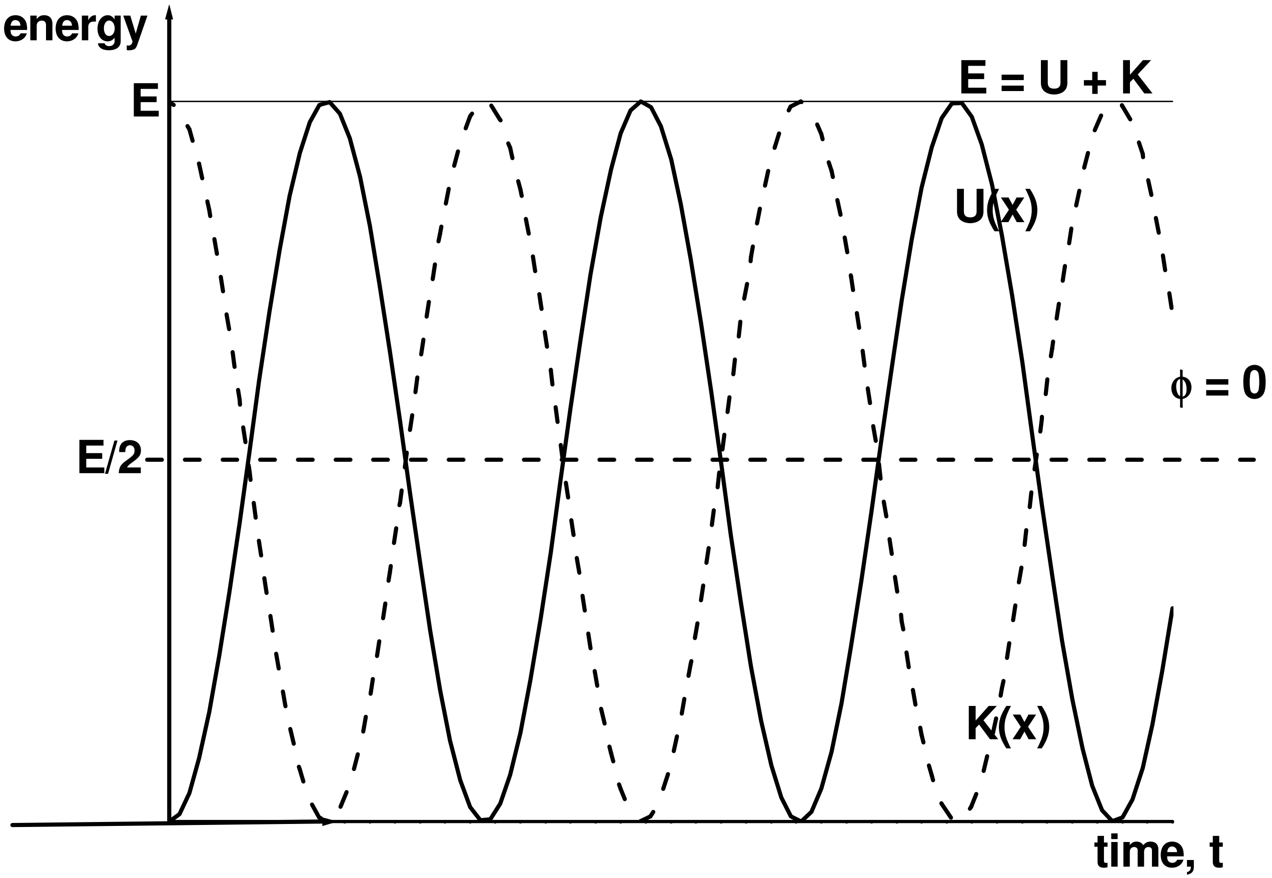
$$\mathbf{= \tfrac{1}{2} kA^2} \quad \underline{\textbf{CONSTANT}}$$

Energy in SHM (cont.)



The energy oscillates between potential energy and kinetic energy as the displacement (x) oscillates between **±A**, the amplitude.
Total energy stays CONSTANT (energy is conserved)

Energy in SHM (cont.)



For SHO the energy also oscillates between **potential** and **kinetic** energy as a function of **time.**

Again note that energy stays CONSTANT

Worked Example

A block of mass 0.2 kg is attached to a spring (k = 15N/m). It is initially held at an extension of 25cm and then released.

Find:

a) the period
b) the energy
c) write an expression for the position x(t)

Answer covered in class

Damped oscillations

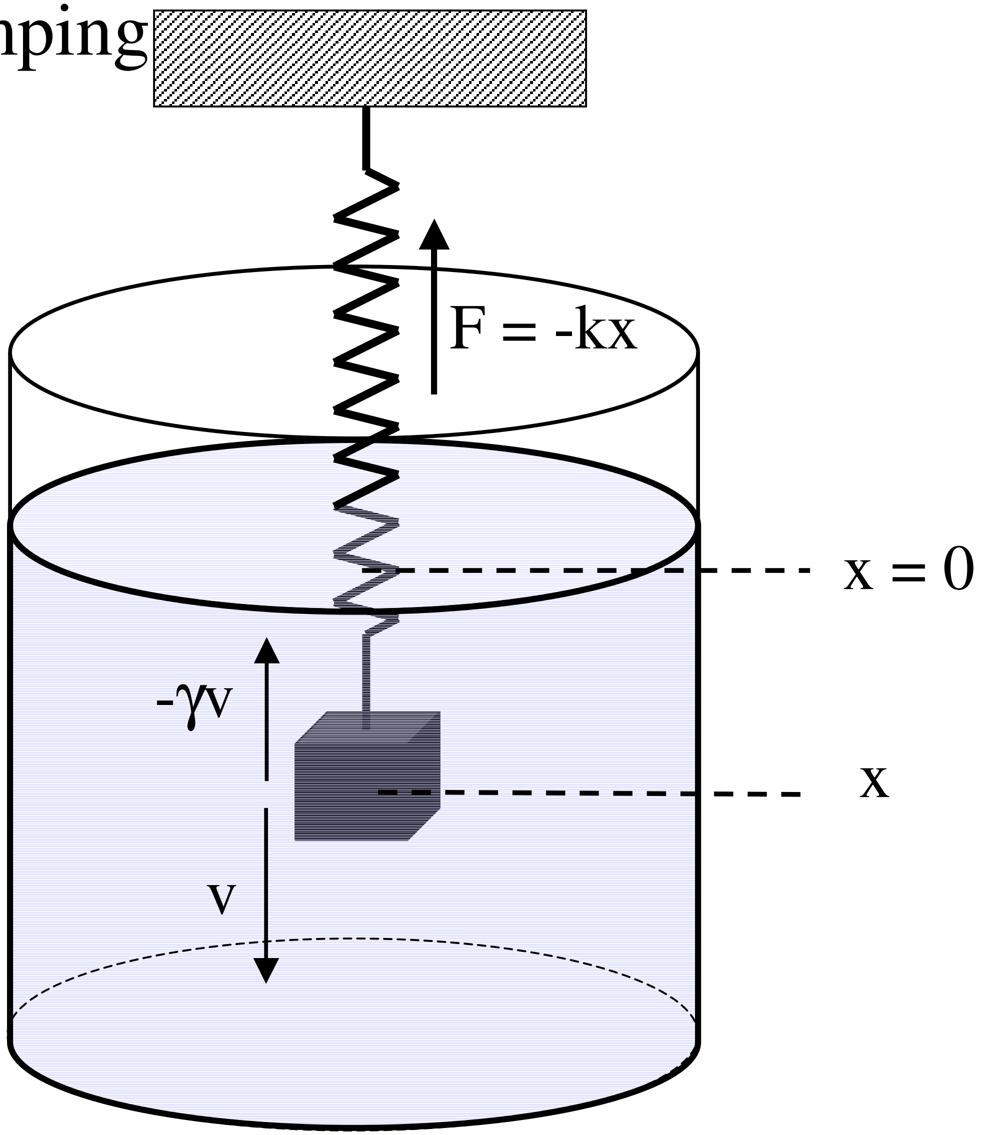
Previously we have looked at the idealised system where there has been no energy loss

In real systems there is always energy loss due to internal/external frictions

In real systems the energy and consequently the amplitude decrease in time

Damped Oscillations - Example

Spring - mass system with damping



Example:
mass oscillating in water.

Water causes a friction force, f.
f = - γv

This force causes a DAMPING of the oscillatory motion.

Damping constant = γ

Damped Oscillations (cont.)

For low velocities resistive force is $\propto$ velocity

$$\underline{f} = -\gamma \underline{v}$$
$$= -\gamma \frac{dx}{dt}$$

γ is the damping constant

Net force on the mass = restoring force + friction force

$$m\frac{d^2x}{dt^2} = -kx - \gamma\frac{dx}{dt}$$

or

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = 0$$

Damped Oscillations (cont.)

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = 0$$ This is the equation of a damped SHO

The equation may be solved and yields:

$$x = A_0 e^{-\gamma t/2m}\cos(\omega' t + \phi)$$

$\omega' =$ Damped angular frequency = $\sqrt{\omega_0^2 - \left(\gamma/2m\right)^2}$

$\omega_0 =$ natural angular frequency = $\sqrt{k/m}$

There are 3 possible cases of damping

1. Underdamping
2. Overdamping
3. Critical damping

Damped Oscillations (cont.) - Underdamping

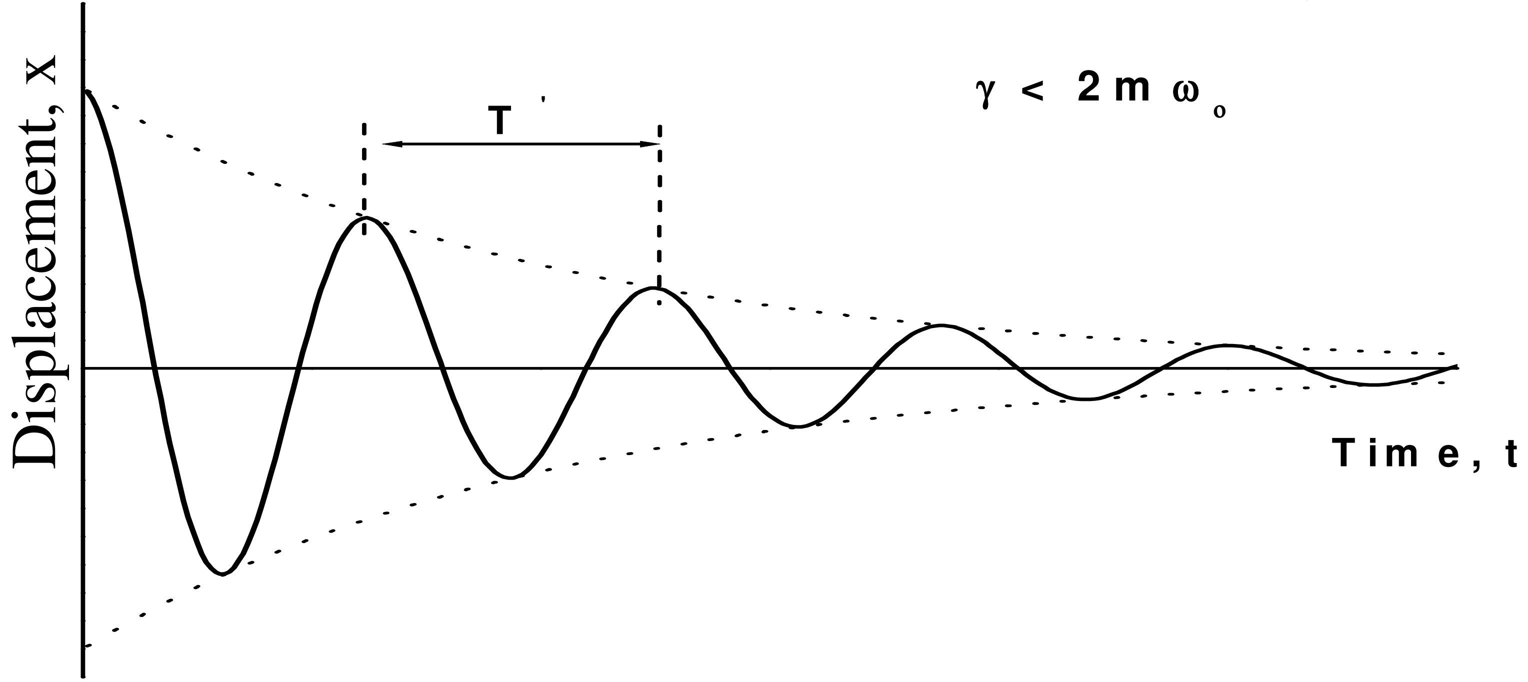
1. $\gamma < 2m\omega_0$ UNDERDAMPING $$\left(\frac{\gamma}{2m}\right)^2 < \omega_0^{\ 2}$$

And ω' is real

The system oscillates with frequency ω' and period T' = 2π/ ω'

This is called **underdamped**

The **amplitude** decays according to: $$A(t) = A_0 e^{\frac{-\gamma t}{2m}}$$



Damped Oscillations (cont.) - Overdamped

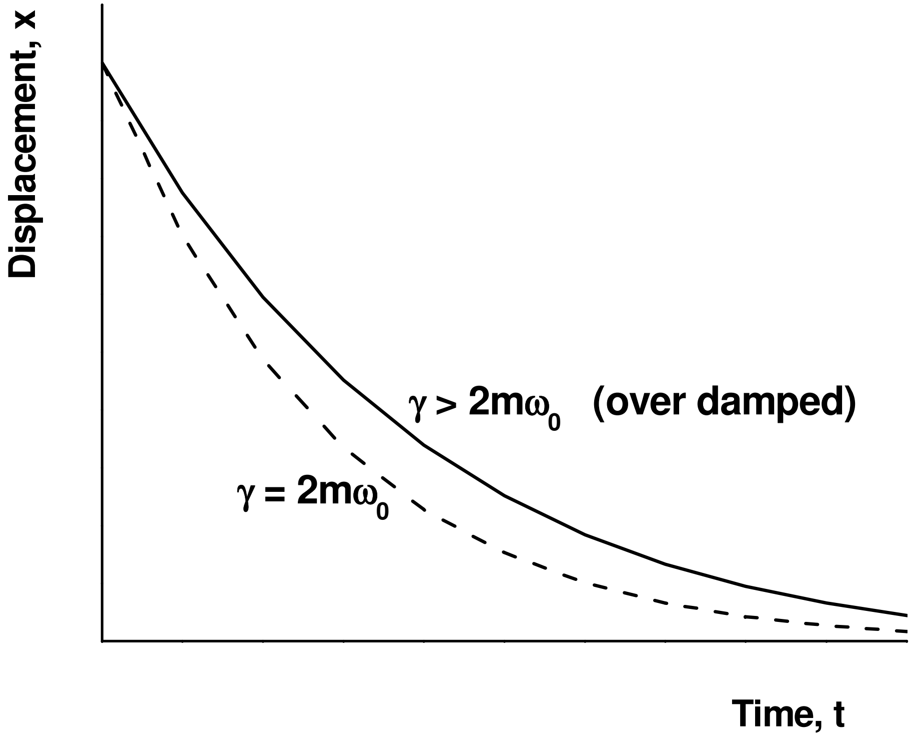
2. $\gamma > 2m\omega_0$ LARGE DAMPING $\left(\frac{\gamma}{2m}\right)^2 > \omega_0^{\,2}$

And $\omega' = \sqrt{-ve\ number}$

= imaginary

In this case there are **no** oscillations

The system moves slowly back to its equilibrium position



Hinged doors that close automatically are overdamped.

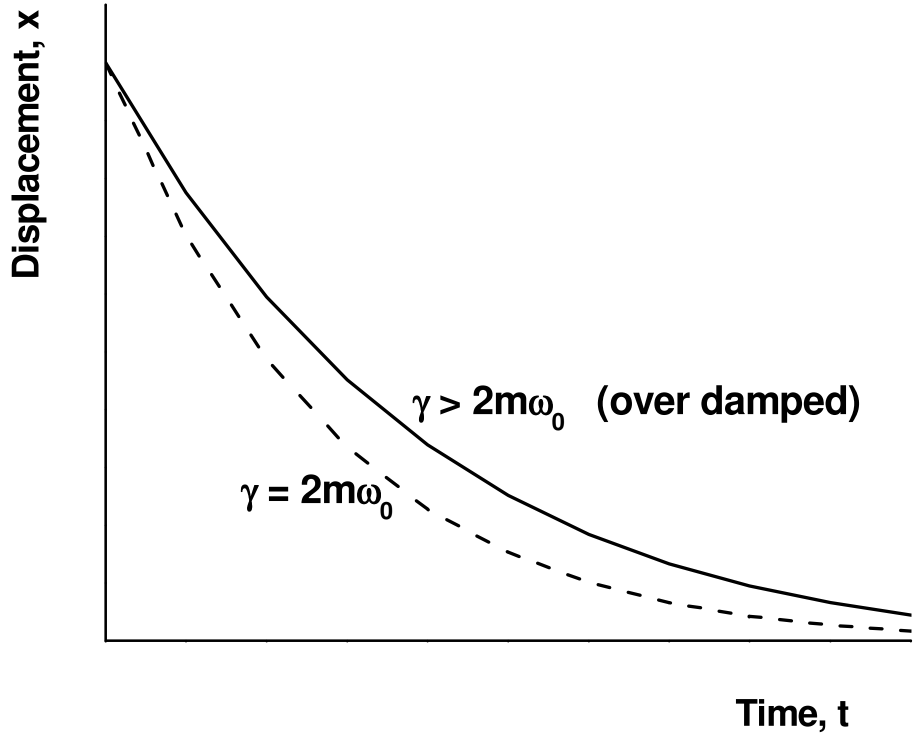
Damped Oscillations (cont.) – Critical damping

3. $\gamma = 2m\omega_0$ CRITICAL DAMPING $\left(\frac{\gamma}{2m}\right)^2 = \omega_0{}^2$

And so ω' = 0

Again there are **no** oscillations

The condition of critical damping leads to the shortest time for the system to return to equilibrium



Critical damping is used in movements of electrical meters to damp the oscillations of the needle. The suspension of a car is adjusted to have somewhat less than critical damping ~ 1.5 oscillations before coming to rest

Forced Oscillations

Loss in energy of a damped oscillator may be compensated by work done by an external agent.

For example a child on a swing

We shall consider an external force that varies sinusoidally at some angular frequency ω_e

$$F(t) = F_0 \cos \omega_e t$$

Equations of motion:

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = F_0 \cos \omega_e t$$

Initially motion is complex.
Ultimately it settles into steady state oscillations. Then energy dissipated by damping is exactly balanced by the external input.

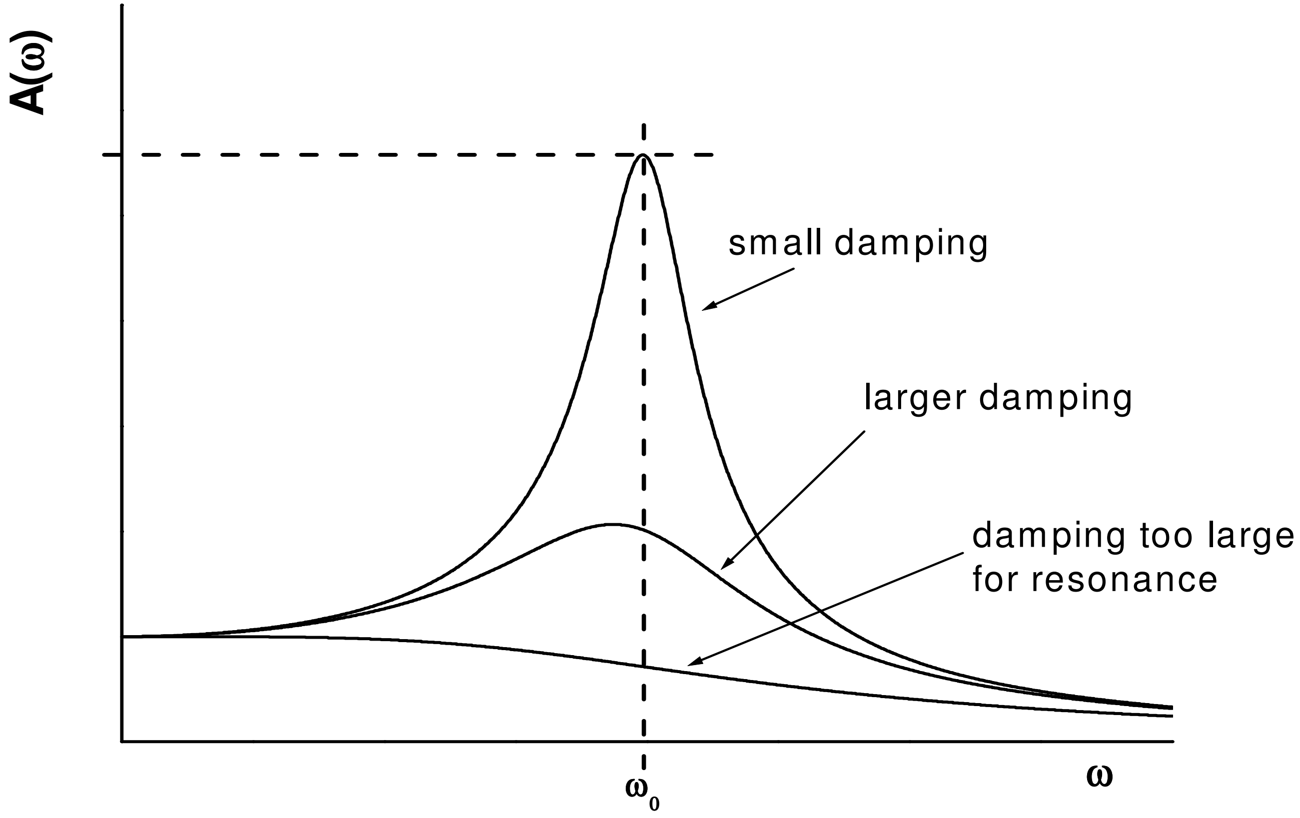
Oscillations have constant amplitude, but the amplitude depends on the external **force frequency**

Forced Oscillations (cont.)

See page 314 of Benson

$$A(\omega_e) = \frac{F_0/m}{\sqrt{(\omega_0{}^2 - \omega_e{}^2)^2 + \left(\frac{\gamma\omega_e}{m}\right)^2}}$$

At $\omega_0 = \omega_e$ the amplitude is at its maximum, RESONANCE



Oscillations in Electrical Circuits

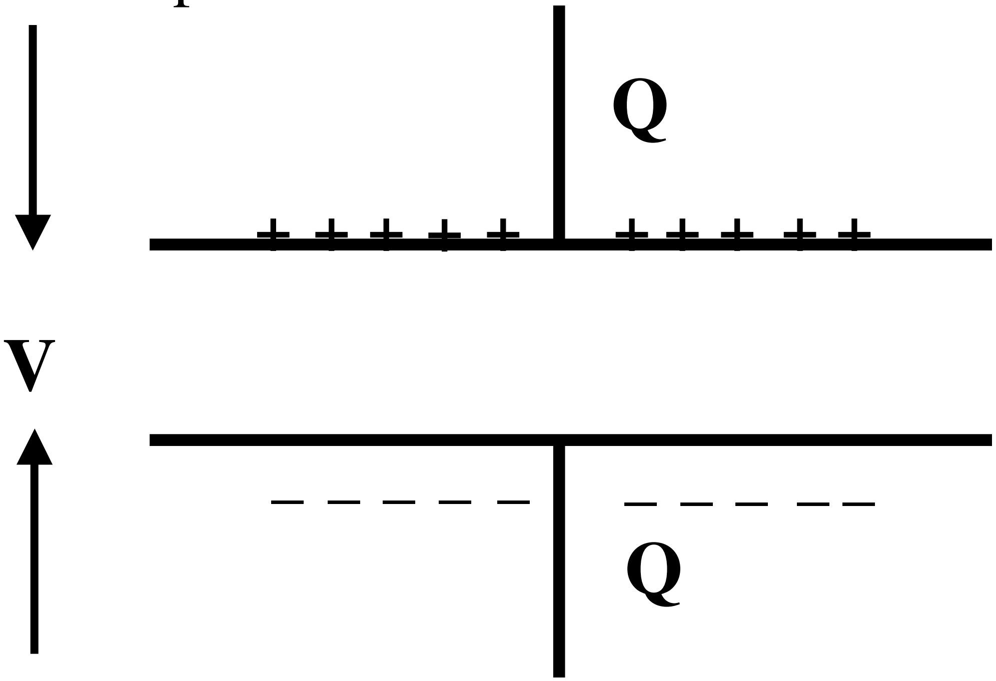
We have shown that mechanical systems can exhibit simple harmonic motion. Now it will be shown that electrical circuits can have this behaviour also

Capacitors and inductors are the important components to these oscillations

So quick revision of these components

Capacitor

When charged to a potential difference V volts, the capacitor holds a charge Q on each plate



Q = CV C = capacitance

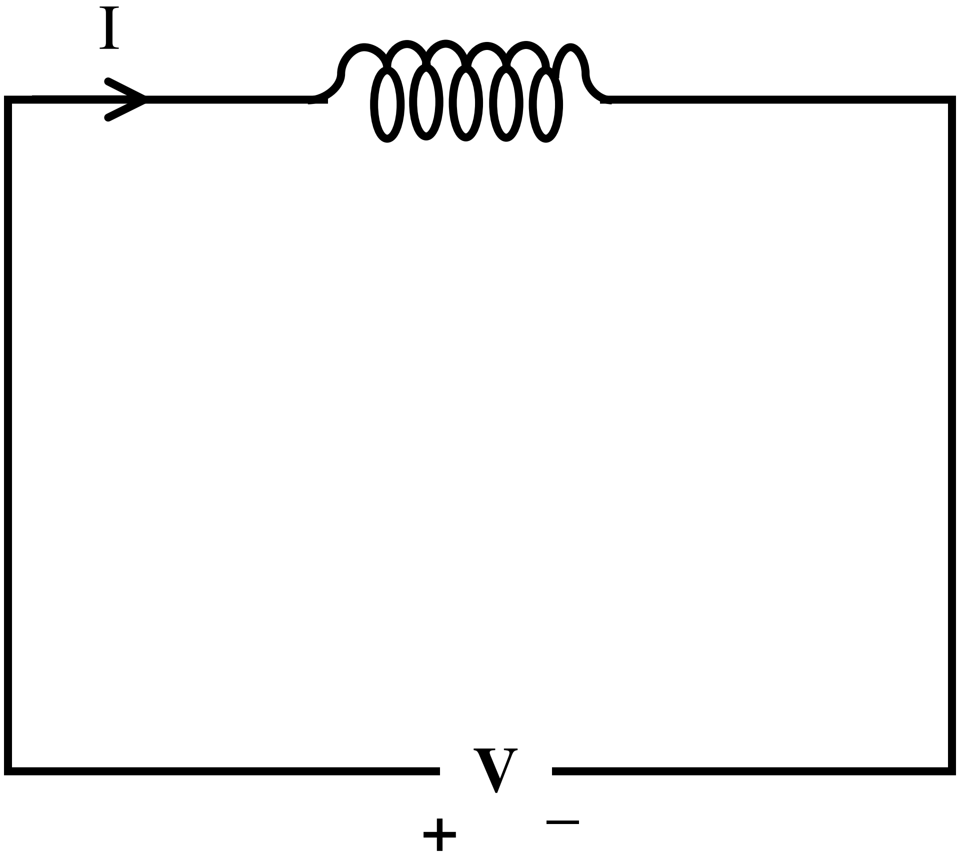
Energy stored in the capacitor $$U = \frac{Q^2}{2C}$$

Inductor

Self inductance = L

Connected to a potential difference of V volts a current builds up in the inductor according to the equation

$$V = L\frac{dI}{dt}$$

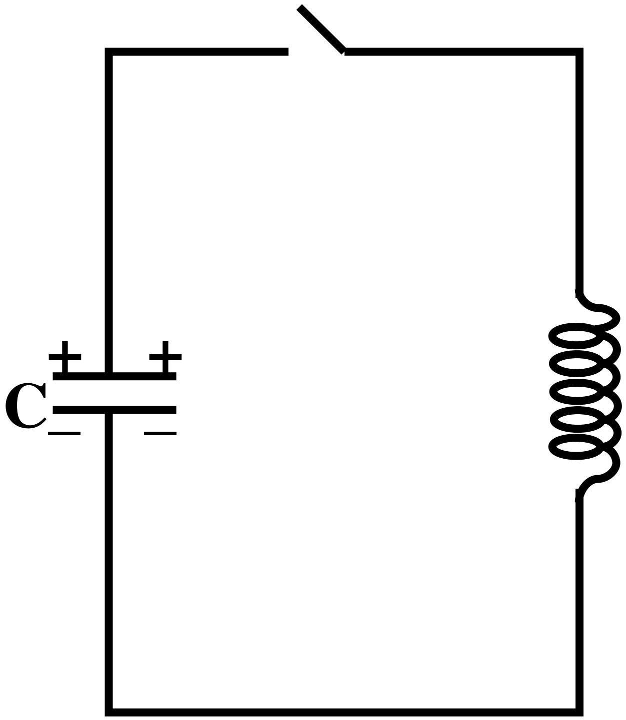


Energy stored in inductor (energy of magnetic field)

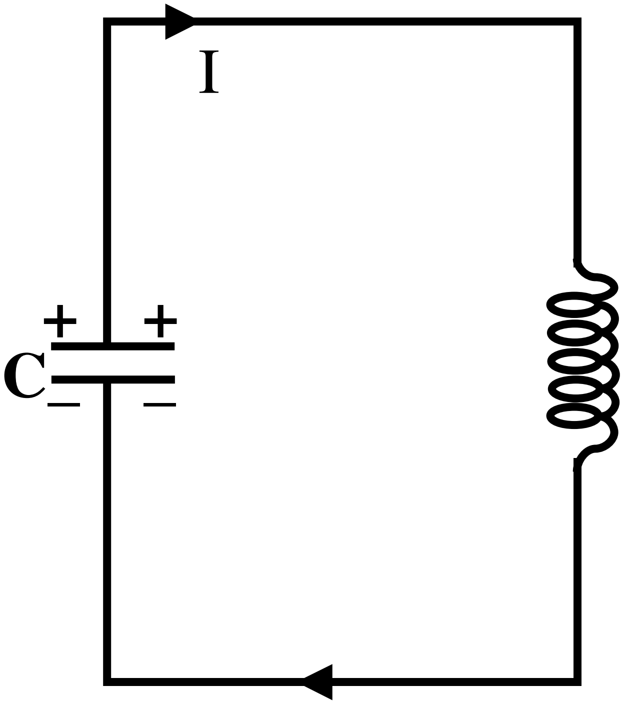
$$U = \frac{1}{2}LI^2$$

LC Oscillations

Capacitor is charged with charge Q on each plate. The switch is open.



At t=0 the switch is closed and and current flows through the inductor



$$V = \frac{Q}{C} = L\frac{dI}{dt}$$

When the switch is closed the current flow causes the charge on the capacitor to decrease

$$I = -\frac{dQ}{dt}$$

Current is the rate of change of charge.
-ve sign indicates the decrease in current

Substitute this into $V = \frac{Q}{C} = L\frac{dI}{dt}$

Get $$\frac{Q}{C} = -L\frac{d^2Q}{dt^2}$$

Rearrange $$\frac{d^2Q}{dt^2} + \frac{1}{LC}Q = 0$$

Above equation has the same form as SHM $\frac{\mathrm{d}^2\mathrm{x}}{\mathrm{dt}^2} + \omega^2\mathrm{x} = 0$

$$\frac{d^2Q}{dt^2}+\frac{1}{LC}Q=0$$

is similar to mechanical SHM

$$\frac{\mathrm{d}^2\mathrm{x}}{\mathrm{d}t^2}+\omega^2\mathrm{x}=0$$

$$\omega_0^2=\frac{1}{LC}$$

The natural angular frequency of the circuit is

$$\omega_0=\frac{1}{\sqrt{LC}}$$

The natural frequency of the circuit is

$$f_0=\frac{\omega_0}{2\pi}=\frac{1}{2\pi\sqrt{LC}}$$

Variation of Q, charge on the capacitor with time is given by

$$Q = Q_0 \sin(\omega_0 t + \phi)$$

At time t=0, charge is maximum on the capacitor
i.e. Q_0

At <u>t=0</u>

$$Q_0 = Q_0 \sin\phi$$

or

$$\phi = \frac{\pi}{2}$$

(sin(π/2) = 1)

$$Q = Q_0 \sin(\omega_0 t + \frac{\pi}{2}) = Q_0 \cos(\omega_0 t)$$

$$Q = Q_0 \cos(\omega_0 t)$$

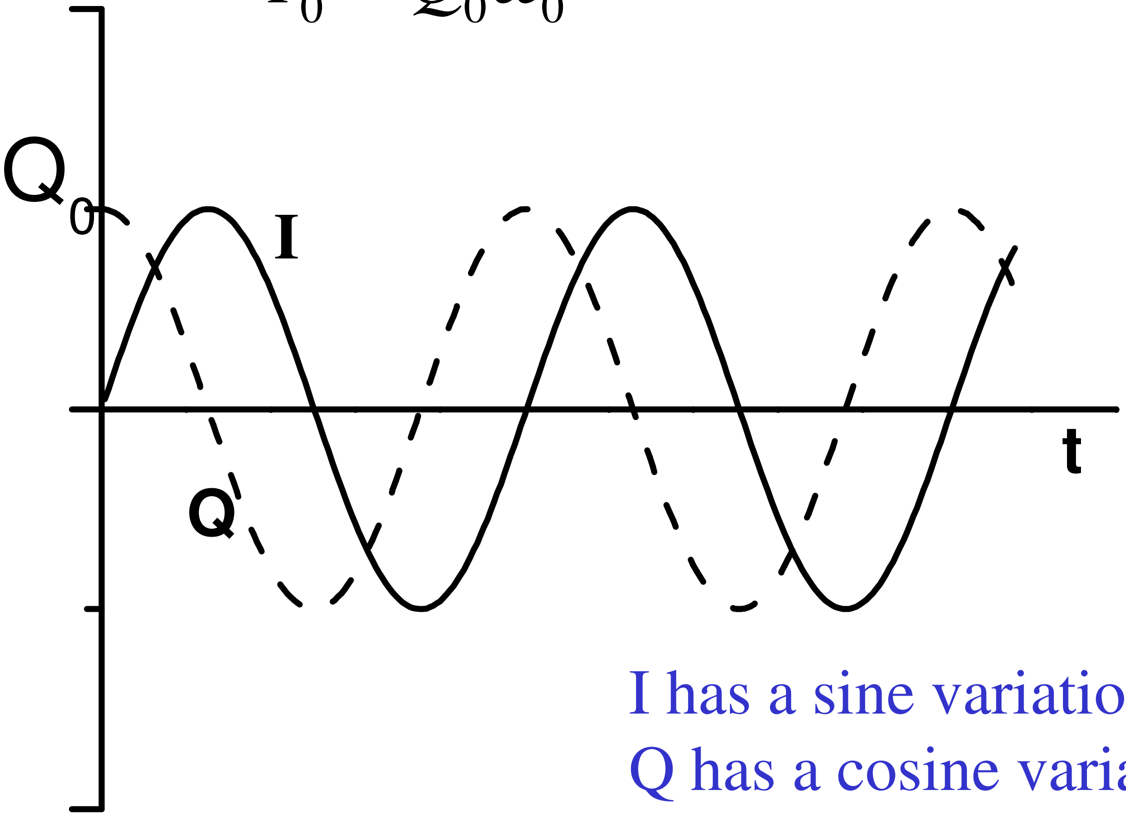
This is how charge varies with time in the LC circuit

The current, I is $-\dfrac{dQ}{dt}$ and $Q = Q_0 \cos(\omega_0 t)$

$$I = Q_0 \omega_0 \sin \omega_0 t$$

$$I = I_0 \sin \omega_0 t$$ where

$$I_0 = Q_0 \omega_0$$



I has a sine variation with time
Q has a cosine variation with time

Analogies between mechanical and electrical quantities

Mechanical	x	v	m	$\frac{1}{2} mv^2$	k	$\frac{1}{2} kx^2$	F	$P=Fv$
Electrical	Q	I	L	$\frac{1}{2} LI^2$	$1/C$	$\frac{1}{2} Q^2/C$	V	$P=VI$

Energy Stored in LC circuit

Lets look at the energy in this LC circuit

Remember $U_E = \dfrac{Q^2}{2C}$ and $U_B = \dfrac{1}{2}LI^2$

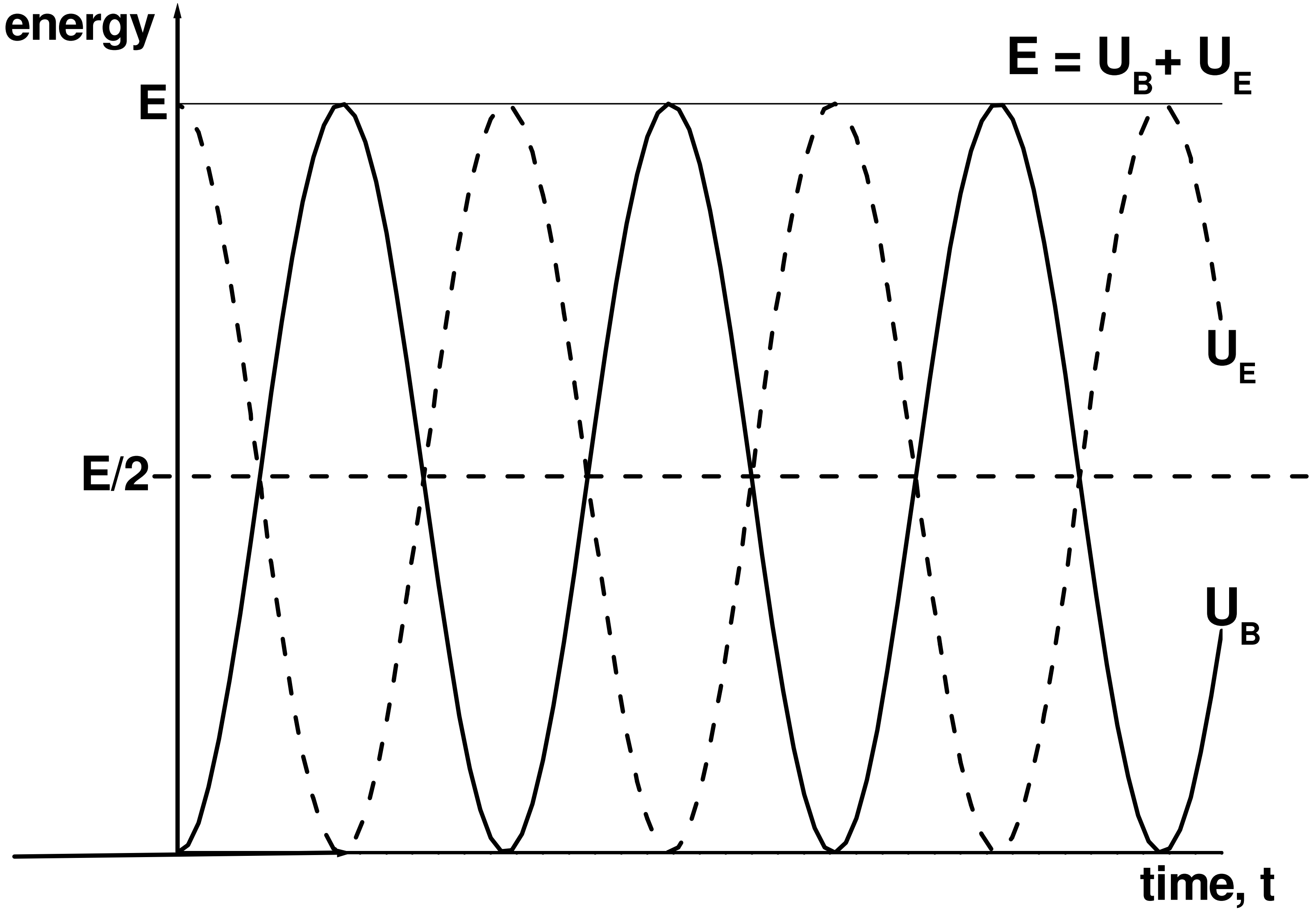
$$U = U_E + U_B = \frac{Q_0^2}{2C}\cos^2\omega_0 t + \frac{LI_0^2}{2}\sin^2\omega_0 t$$

since $\omega_0^2 = \dfrac{1}{LC}$ and $I_0 = Q_0\omega_0$

$$U = \frac{Q_0^2}{2C}\cos^2\omega_0 t + \frac{1}{\omega_0^2 C}\frac{\omega_0^2 Q_0^2}{2}\sin^2\omega_0 t$$

$$= \frac{Q_0^2}{2C}(\cos^2\omega_0 t + \sin^2\omega_0 t) = \frac{Q_0^2}{2C} = \frac{1}{2}LI_0^2$$

= CONSTANT



There is an exchange of energy between the capacitor and the inductor

What is happening physically in the LC circuit?

Page 659 Benson

All energy in capacitor

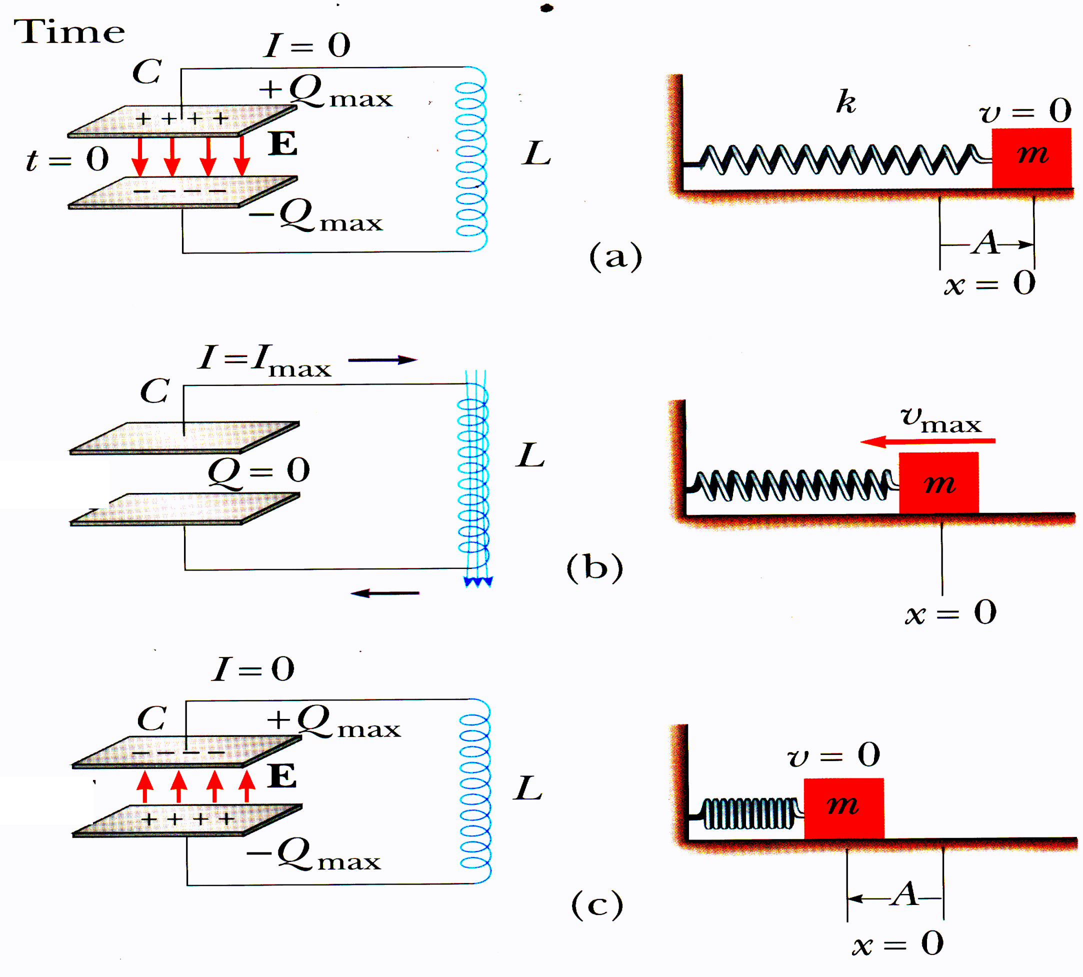
I=0 when Q=Q_0

All energy in inductor

Q=0 when I=I_0

All energy in capacitor

But polarity is opposite to initial case



Serway, *Overhead Transparencies t/a Physics for Scientists & Engineers*, 4/e
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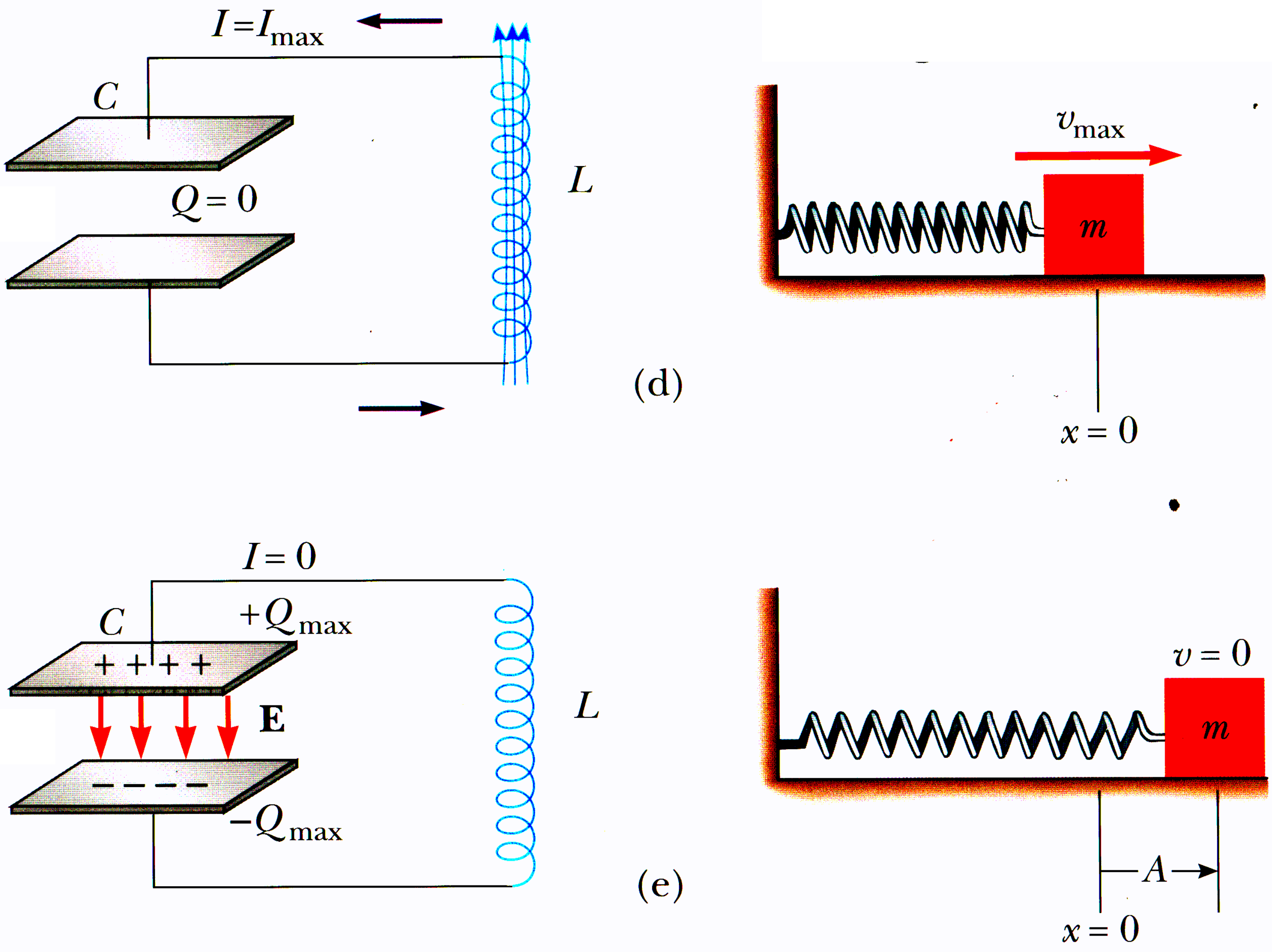
Raymond A. Serway



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Raymond A. Serway

Worked Example

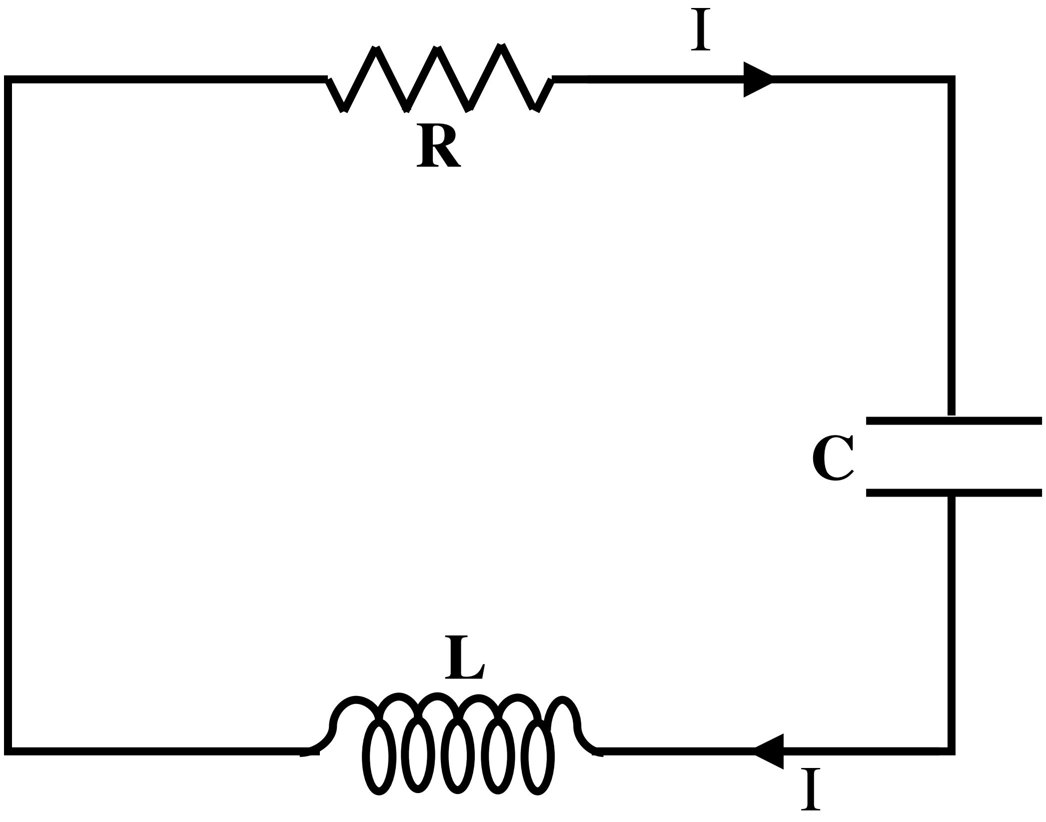
A capacitor, C= 5 x 10^{-6} F is connected to an inductor, L = 4 mH. See figure for circuit. The initial potential difference across the capacitor is 20V. At time t = 0s the switch is closed. Find:

a) The maximum charge Q_o on the capacitor. (2)
b) The maximum energy stored in the inductor. (3)
c) The natural angular frequency. (2)
d) The current when $Q= Q_o/2$ (5)
e) Write down the expressions for Q(t), I(t) (4)

Damped LC oscillations

LCR Circuit

Resistance in a circuit causes energy dissipation (P=I^2R and P=E/t)



Remove the battery and close the switch. Current begins to flow round circuit

The LC part of the circuit means the energy is oscillating between electrical (capacitor) and magnetic (inductor), **BUT** the resistor loses some of this energy over time

So the LCR circuit gives rise to DAMPED OSCILLATIONS

$$\overbrace{\frac{Q}{C}}^{\text{Voltage across capacitor}} - \underbrace{IR}_{\text{Voltage across resistor}} - \overbrace{L\frac{dI}{dt}}^{\text{Voltage across inductor}} = 0$$

Remember $I = -\frac{dQ}{dt}$ Substitute into top equation

Gives $$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

$$L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{Q}{C}=0$$

Has same form as

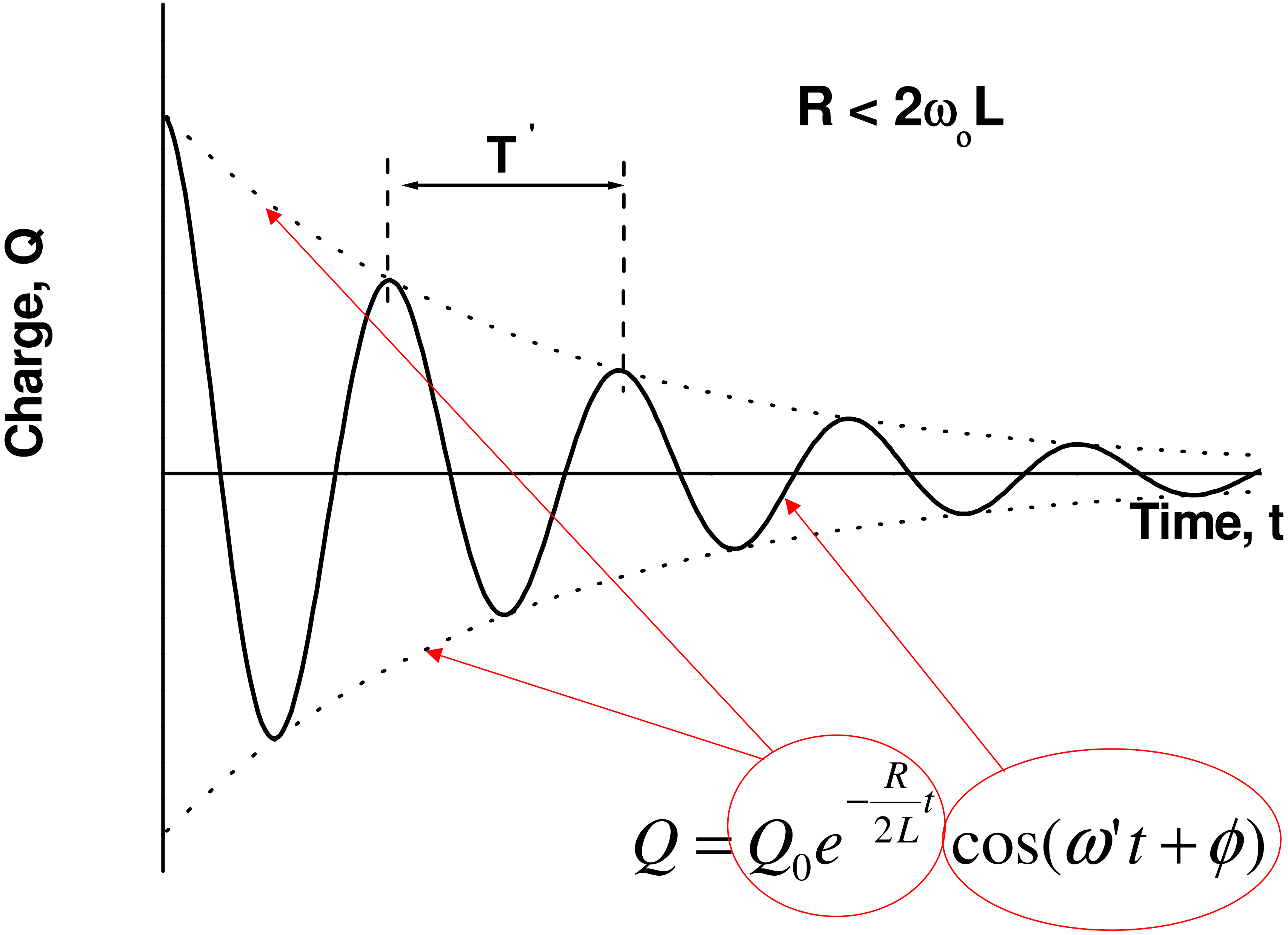
$$m\frac{d^2x}{dt^2}+\gamma\frac{dx}{dt}+kx=0$$

Therefore solution can be written as:

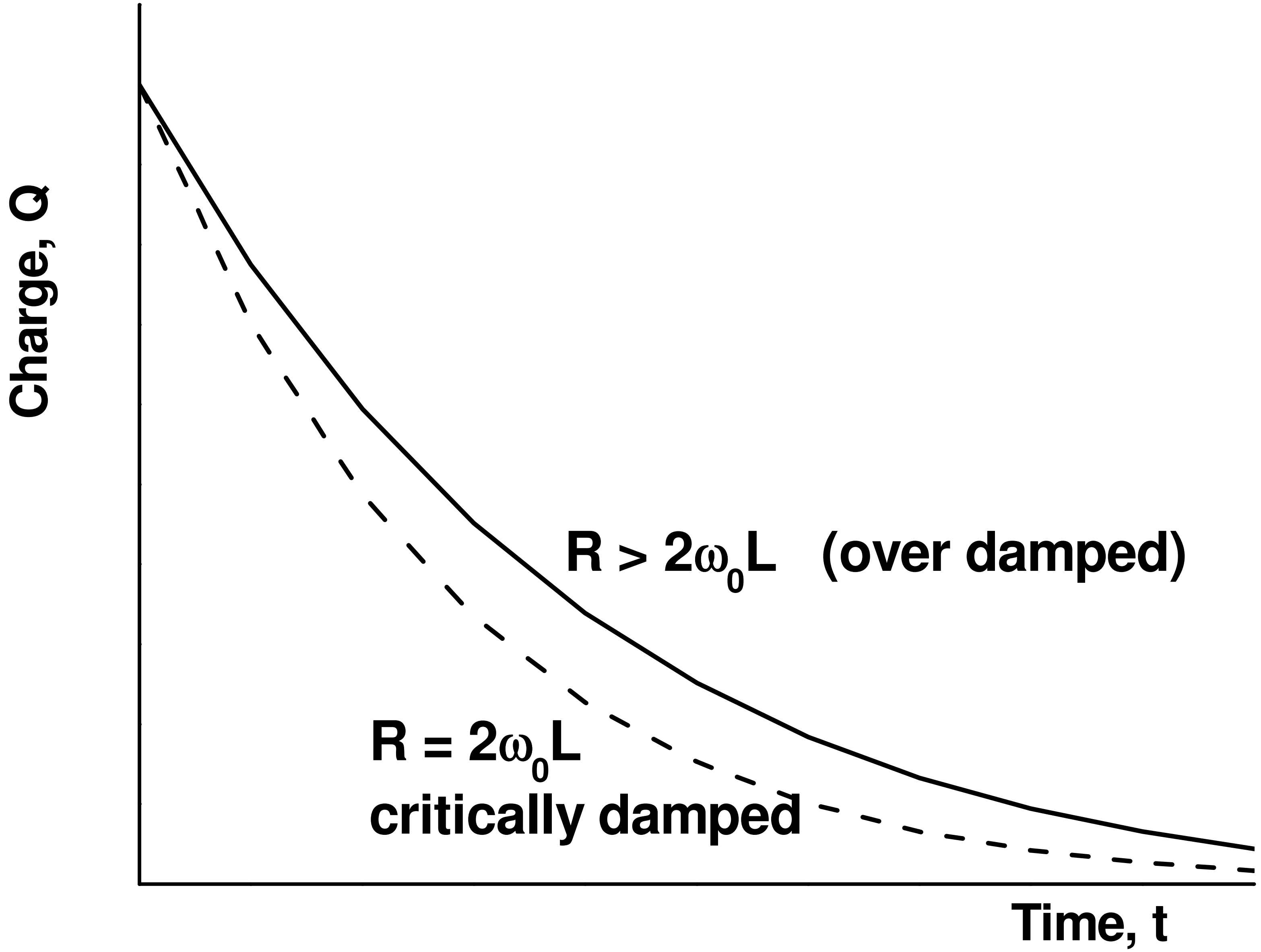
$$Q=Q_0e^{-\frac{R}{2L}t}\cos(\omega' t+\phi)$$

Where damped frequency ω’ is

$$\omega'=\sqrt{\omega_0^2-\left(\frac{R}{2L}\right)^2}$$



$$Q = Q_0 e^{-\frac{R}{2L}t}\cos(\omega' t + \phi)$$


Charge, Q
R > 2ω_0L (over damped)
R = 2ω_0L
critically damped
Time, t

- Slides will be made available on EE1 website (http://www.physics.gla.ac.uk/EE1)