EE1 Electrostatics: Slide Notes

Textbook: Benson, Chs. 22 - 26

Electrostatic Force

Benson 22.1

It is observed experimentally that a force exists between material objects that have the property of electric charge.

This is called the electromagnetic force.

It is the existence of this force that shows the existence of charge as one of the fundamental properties of matter along with size and mass.

The electromagnetic force between charges is one of the fundamental forces of nature like the gravitational force between masses.

The electromagnetic force has two component parts:

The electrostatic force which exists between stationary or “static” charges

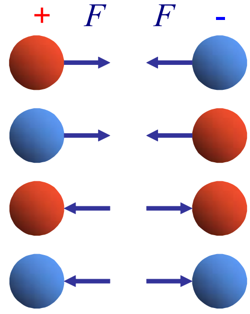
The magnetic force which exists between moving charges or electric currents

It is found experimentally that charge exists in two states which are called positive charge and negative charge

The difference of charge states is seen through the difference in the force behaviour between charged objects

Attractive forces between charges of opposite sign

Repulsive forces between charges of the same sign



Charge originates at the fundamental particle level

For the atomic particles forming stable matter have

Proton = +1 unit of charge

Electron = -1 unit of charge

Neutron = 0 charge

All positive and negative fundamental particle charges have exactly the same magnitude

e.g. atom with Z protons and (A-Z) neutrons in the nucleus surrounded by Z electrons is electrically neutral

For the fundamental particles charge is said to be “quantised”

The “quantum” of charge is very small = 1.6×10^{19} coulomb

In the “classical limit” we treat charge as being continuous

Charge is conserved

Benson 22.2

For the study of electrostatics we divide materials into two classes:

Conductors which allow charge to move freely through them

(metals and carbon in which the outermost atomic electrons are “free” to move through the crystaline atomic lattice of the material)

Insulators in which charge does not move but stays where it is placed

(glass, quartz, mica, rubber, plastics in which the atomic electrons are tightly bound to their nuclei and are not free to move)

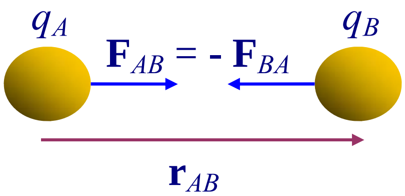
We will assume both classes of materials to be “perfect”:

In conductors any charge will move with complete freedom

In insulators there will be no charge or any applied charge will be “fixed”

Benson 22.5

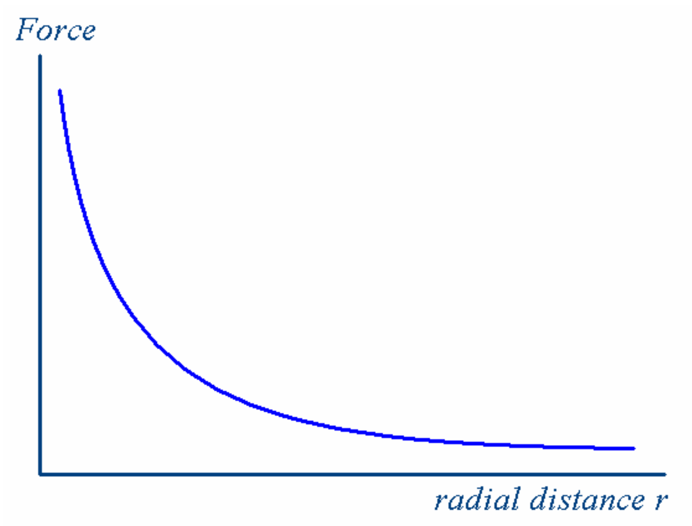
The electrostatic force is described by the **Coulomb Force Law**



For point or spherical charges q_A, q_B separated by distance $\mathbf{r}_{AB}$ between the charge centres the force on charge q_B due to charge q_A is

$$\mathbf{F}_{BA} = \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{q_A q_B}{r^2}\hat{\mathbf{r}}_{AB} \quad \text{with} \quad \mathbf{r}_{AB} = r\hat{\mathbf{r}}_{AB}$$

- The magnitude of the force is proportional to the product of the charge values
- The direction of the force is along the line joining the charge centres in the direction of $\mathbf{r}_{AB}$ if the charge product is positive, oppositely directed if the product is negative.
- By Newton's Second Law both charges experience equal magnitude but oppositely directed forces.
- The Coulomb electrostatic force is an inverse-square law force since it depends on $1/r^2$.



- The $1/4\pi$ factor is used with the $1/r^2$ behaviour to “rationalise” or simplify many theoretical electromagnetic relations. (See Gauss’ Law later)
- The constant ε_0 is the permittivity of free space or vacuum permittivity.
- In the Systeme Internationale (SI) system of units it is defined to be 8.854×10^{-12} $C^2N^{-1}m^{-2}$ or Fm^{-1}.
- This definition uses the speed of electromagnetic waves in vacuum, c.

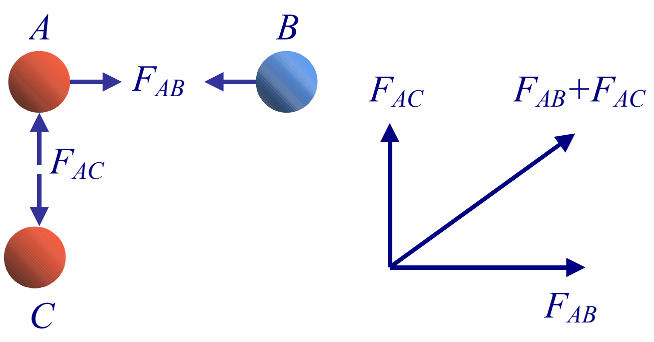
Ampere’s law defines the unit of current, hence charge, in terms of force, thus m, l, t: $C^2\mu_0 \propto ml$

Coulomb’s law relates charge to force: $C^2/\varepsilon_0 \propto ml^3/t^2$

Ampere’s law + Coulomb’s law $\rightarrow$ Maxwell’s equations: $c^2 = 1/(\mu_0\varepsilon_0)$

ratio of two charge relations above: $\mu_0\varepsilon_0 \propto t^2/l^2 \propto 1/c^2$

Individual forces between charges add linearly as vectors - the superposition principle



Ex: Charge A = 20μC, charge B = -15μC, charge C = 25μC, distance AB = 0.1m, distance AC = 0.07m.

Ans: 957.2 N at 73.6°

Electric Field

Benson Ch.23

The electric field concept provides a very useful *description* of the electrostatic force that assists visualisation and calculation.

The electric field is a way of "mapping" the electric force.

Electric charge is regarded as the "source" and "sink" of the electric field which is thought of as filling all space around charges.

The strength or intensity and direction of the electric field is given by the electric field strength vector $\mathbf{E}$ at the field point.

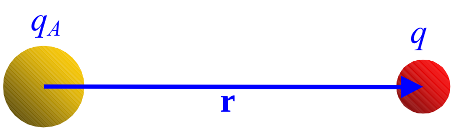
Electric field strength is the force per unit charge on a point charge at the field point. For a "test" charge *q* at the field point

$$\mathbf{E} = \frac{\mathbf{F}}{q}$$

$\mathbf{E}$ depends on the field source charges and does not depend on charge *q*.

Units of electric field strength are NC^{-1}

Apply Coulomb's Law to the test charge *q* in the field of a point source charge q_A:

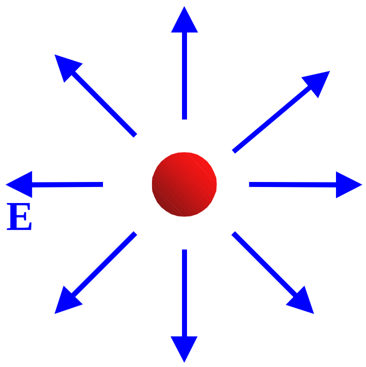


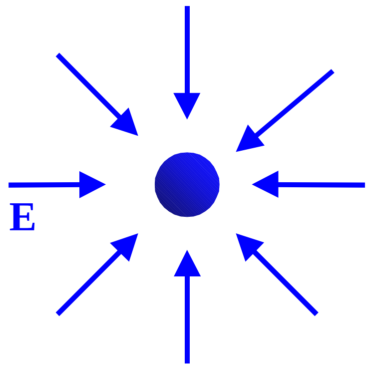
$$\mathbf{F}_q = \frac{q q_A \hat{\mathbf{r}}}{4\pi\varepsilon_0 r^2}$$

Electric field of a point charge is then

$$\mathbf{E}(\mathbf{r}) = \frac{q_A \hat{\mathbf{r}}}{4\pi\varepsilon_0 r^2}$$

E has the same value in all directions for a given *r* value, so the field is





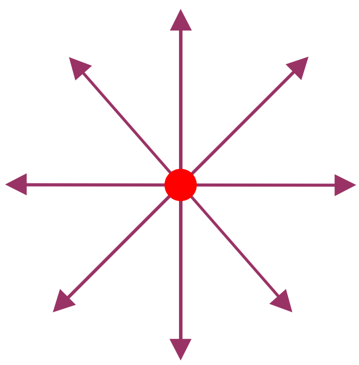
symmetrical. If the charge sign is reversed the field direction is reversed.

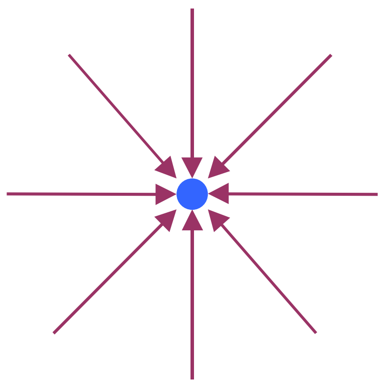
The form of an electric field can be conveniently displayed using “lines of force” which “map” the field.

The conventions for this field visualisation technique are:

- the direction of a line of force is the same as the direction of the field at that point,
- the density of lines per unit area is proportional to the magnitude of the field.

Isolated point charges acting as sources and sinks of the field have lines of force of the form:





Ex: Electric fields of two equal magnitude point charges.

Sketch the form of the field for (a) two positive charges and (b) for one positive and one negative charge.

Obtain an expression for the electric field in the plane mid-way between the charges perpendicular to the line joining them:

$$E_{rad} = \frac{2Qs}{4\pi\varepsilon_0 (s^2 + (d/2)^2)^{3/2}}$$

Find the maximum field in this plane: s = ±d/4

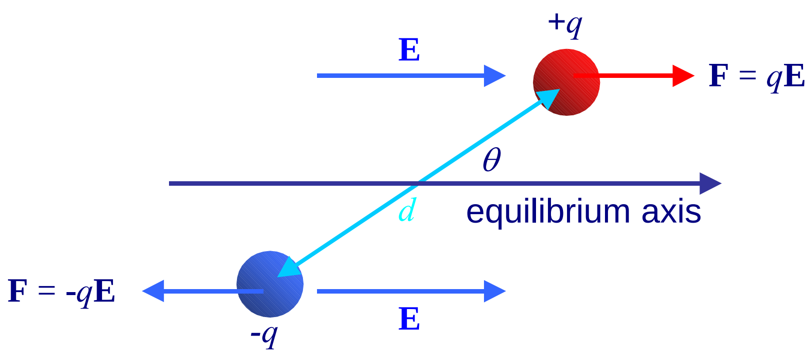
Ex: Find the electric field due to an infinite line charge (Benson Ex. 23.7):

$$E_{rad} = \frac{\sigma}{2\pi\varepsilon_0 d}$$

Electric Dipole

Benson 23.6

A system of two mechanically coupled equal but opposite sign charges forms an **electric dipole**.



Placed in a uniform electric field the dipole charges experience equal but oppositely directed forces qE.

These forces produce a torque on the dipole

$$\Gamma = 2 \times qE \times \frac{d}{2} \times \sin\theta \;= qdE\sin\theta$$

The product qd is the **dipole moment**.

The torque tends to align the dipole along the field direction.

If the dipole is rotated against this torque away from its equilibrium alignment along the field direction work must be done.

The means of rotating the dipole puts energy into the dipole-field system to be stored as potential energy.

For a small angular rotation $d\theta$ the work done is $dW = \Gamma d\theta$

For rotation through angle θ the work done storing energy is

$$W(\theta) = \int_0^\theta \Gamma d\theta = qdE\int_0^\theta \sin\theta d\theta = qdE\left[-\cos\theta\right]_0^\theta = qdE(1-\cos\theta)$$

The maximum value of this energy is when

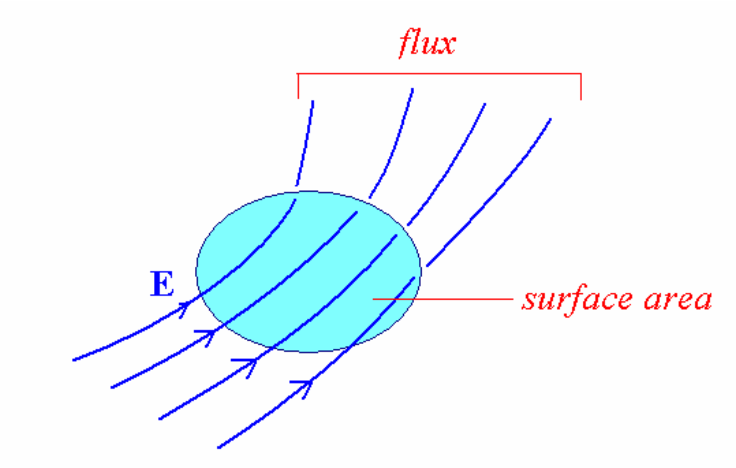
$$\cos\theta = -1,\ \ \theta = \pi \ \ \text{(180° rotation) and } W_{\max} = 2qdE$$

Gauss' Law

Benson Ch. 24

The **amount** of electric field passing through a surface is the **electric flux** through that surface. (Flux is a property of any vector field.)

Using lines of force to represent the field we have

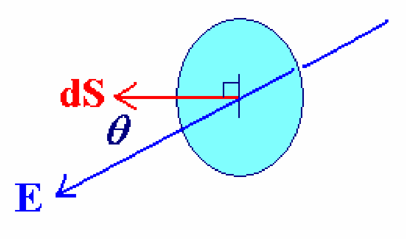


Since the electric field comes from / goes to conserved charges the field must be conserved.

If there are no charges inside a closed surface the flux entering the surface must the equal the flux leaving the surface.

The **electric flux** of an electric field **E** through an element of area **dS** is

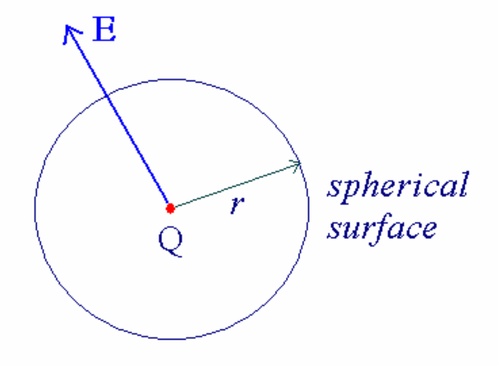
$$d\phi = \mathbf{E.dS} \equiv EdS\cos\theta$$



Over a finite surface this is $\phi = \int_S \mathbf{E.dS} = \int_S E\cos\theta dS$

Flux ϕ is a scalar quantity.

Ex: Flux and a cylindrical surface parallel to a uniform electric field



The electric field passing through spherical surface centred on a point charge Q is

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

This field is everywhere normal to the surface so that the electric flux through the surface is

$$\phi = E \times 4\pi r^2 = \frac{4\pi r^2 Q}{4\pi\varepsilon_0 r^2} = \frac{Q}{\varepsilon_0}$$

Because of the inverse square-law form of Coulomb's law the result is independent of the size of the spherical surface.

A general analysis shows that the result is also

- independent of the shape of the enclosing surface

- independent of the position of the charge within the surface

If there are several charges Q_1, Q_2, … within the surface the resultant electric flux is then

$$\phi = \frac{Q_1 + Q_2 + \ldots}{\varepsilon_0}$$

Note that negative charges produce a negative, inwardly directed flux component so that if the **net charge** within the surface is

$$\sum_i Q_i \quad \text{then} \quad \phi = \sum_i \frac{Q_i}{\varepsilon_0}$$

Using the definition of electric flux then

$$\boxed{\oint_S \mathbf{E}.d\mathbf{S} = \frac{\sum_i Q_i}{\varepsilon_0}}$$

This is **Gauss' Law**

It is a consequence of the inverse square-law behaviour of Coulomb's Law.

S is called a Gaussian surface. Note that it is a *closed* surface.

Note the net sum of all the charges within the Gaussian surface.

Gauss' Law can be applied where the *form* of the electric field can be deduced from the *symmetry* of the arrangement of charges and conducting surfaces. For this type of situation it is simpler to use than Coulomb's law.

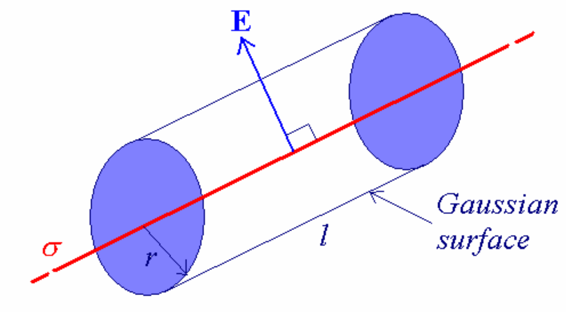
The closed Gaussian surface is usually formed by surfaces that are

- perpendicular to the electric field so that on the surface **E.dS** = *EdS*

- parallel to the electric field so that on the surface **E.dS** = 0

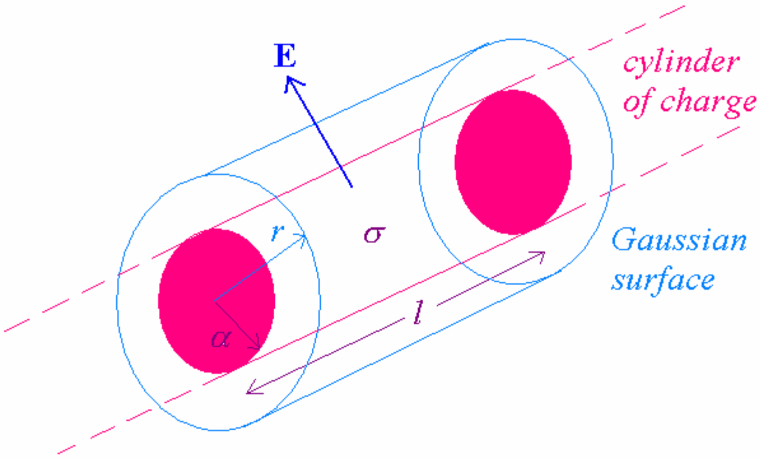
The spherical surface used above is an example of a surface where the field is everywhere perpendicular to it.

Ex: Field of an infinitely long line charge



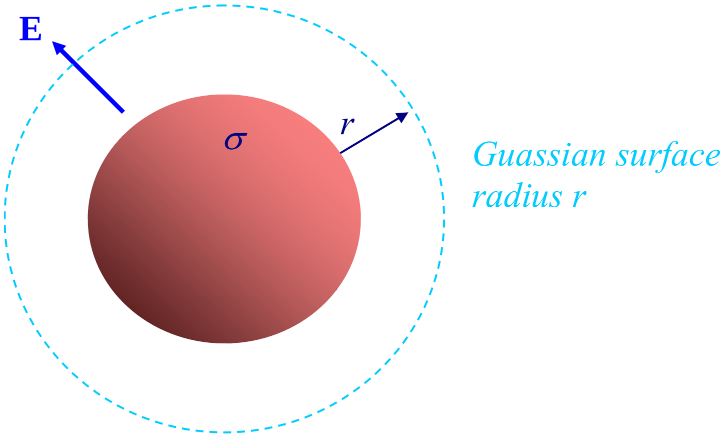
$$E_{rad} = \frac{\sigma}{2\pi\varepsilon_0 d}$$

Ex: Field of a cylinder of charge



$$E = \frac{\sigma a^2}{2\varepsilon_0 r} \text{ for } r \geq a \text{ and } E = \frac{\sigma r}{2\varepsilon_0} \text{ for } r \leq a$$

Ex: Field of a sphere of uniform charge density



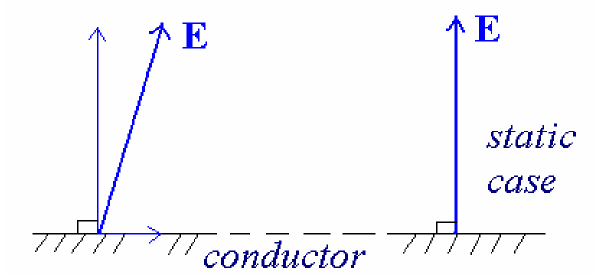
$$E = \frac{a^3\sigma}{3\varepsilon_0 r^2} \text{ for } r \geq a \text{ and } E = \frac{\sigma r}{3\varepsilon_0} \text{ for } r \leq a$$

Electric field and conductors

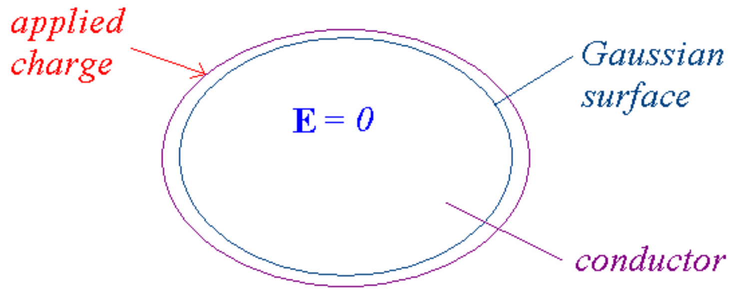
Benson 23.4

If an electric field exists inside a conductor it will cause the conduction charges to move (electric currents) and there will not be a static charge situation – for the electrostatic case the electric field inside a conductor must be zero.

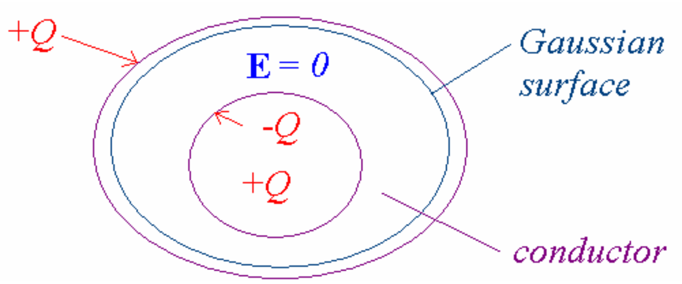
If an electric field is at an angle to a conductor surface which is not perpendicular to the surface the component of the field parallel to the surface will cause charge movement – for the electrostatic case the electric field must be perpendicular to a conductor surface.



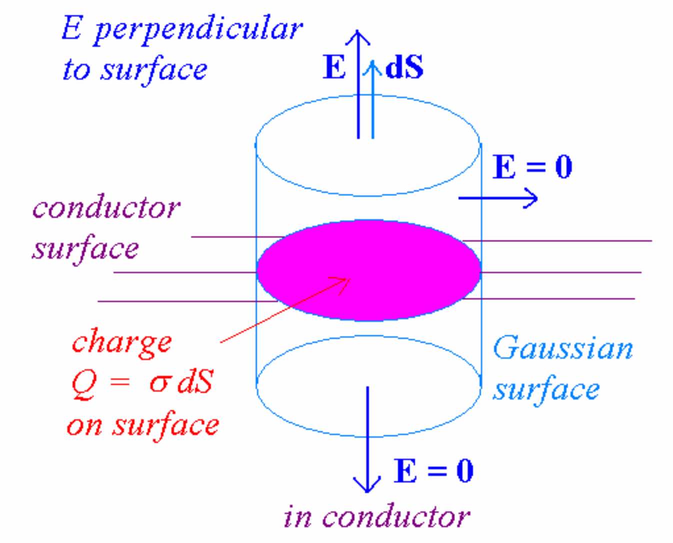
Apply a charge to a conductor



Since **E** is zero inside the conductor the flux through the Gaussian surface is zero and any applied charge must be on the surface of the conductor.



Place charge within a hollow conductor. The flux within the conductor is zero so that +Q induces –Q on the inner surface inducing +Q on the outer surface



A cylindrical Gaussian surface intersects the surface of a charged conductor; the enclosed charge is σdS.

$$\oint_S \mathbf{E}.\mathbf{dS} = EdS = \frac{Q}{\varepsilon_0} = \frac{\sigma dS}{\varepsilon_0} \rightarrow E = \frac{\sigma}{\varepsilon_0}$$

Ex: If electrical breakdown (arcing) occurs in dry air at a field strength of 3 MVm^{-1} what is the maximum charge that can be put on a wire 2 mm in diameter and 2 m long?

Ans: 0.334μC

Electrostatic energy

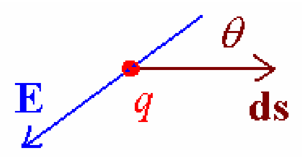
Benson Ch. 25

Physical systems can usually be described in terms of either the **forces** acting within a system or, equivalently, the **energy** behaviour of a system.

In many cases it is easier to analyse the energy behaviour rather than the force behaviour since energy is a scalar quantity.

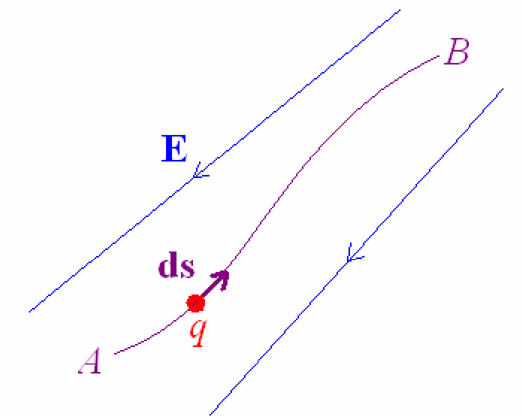
The electric field strength at a point is defined by the force acting on a charge q: $\mathbf{F} = q\mathbf{E}$

If this charge is moved **against** the force the **work** that is **done on the charge** to move it a distance **ds** is: $dW = -q\mathbf{E}.\mathbf{ds} = -qE\cos\theta ds$



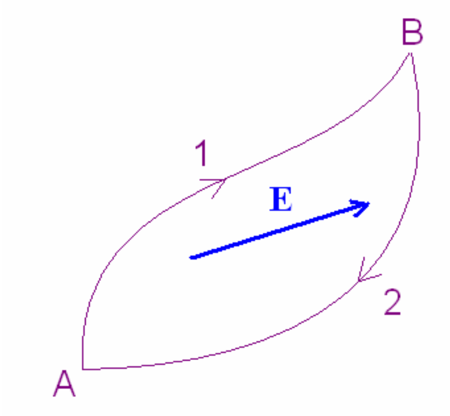
E.ds is the scalar product of the vectors **E** and **ds** = $E_x dx + E_y dy + E_z dz$

This work on the charge **raises its potential energy** by dW



If the charge is moved a finite distance through the electric field the total work done against the field force and stored as potential energy by the charge will be the sum of all the elemental energy increments dW for small charge movements **ds**.

$$W_{AB} = \lim_{ds \to 0} \sum_{AB} dW = \lim_{ds \to 0} \sum_{AB} -q\mathbf{E}.\mathbf{ds} \equiv -q\int_A^B \mathbf{E}.\mathbf{ds}$$



If a charge is moved from A to B along path 1 let it change energy by W_1.
Along path 2 from B to A let it change energy by W_2.

If $W_2 \neq W_1$ then the charge could be constantly moved around the loop gaining energy - violating the **principle of conservation of energy**.

The energy change in the field between A and B must be the same for all paths so **the energy change is independent of the path and depends only on the initial and final positions**.

Electric potential

Benson 25.1

At an infinite distance away from any field source charges the electric field will go to zero and the potential energy of a test charge can be considered zero.

If the work done in moving a charge q from infinity to a field point A is W_A then the **potential** at A is

$$V_A = \frac{W_A}{q}$$

If the charge is moved to a point B with total energy change W_B ,

$$V_B = \frac{W_B}{q}$$

By conservation of energy the work done in moving between A and B is

$$W_{AB} = W_B - W_A$$

The **potential difference** is then

$$V_{AB} = \frac{W_B - W_A}{q} = V_B - V_A$$

The work done in moving charge q through an electric field is

$$W_{AB} = -q\int_A^B \mathbf{E}.\mathbf{ds}$$

From the definition of potential difference then

$$V_{AB} = -\int_A^B \mathbf{E}.\mathbf{ds}$$

The scalar product of the field E and the positional displacement **ds** is

$$\begin{aligned}\mathbf{E}.\mathbf{ds} &= (\mathbf{i}E_x + \mathbf{j}E_y + \mathbf{k}E_z).(\mathbf{i}dx + \mathbf{j}dy + \mathbf{k}dz) \\ &= E_x dx + E_y dy + E_z dz\end{aligned}$$

If A is the point x_A, y_A, z_A and B is the point x_B, y_B, z_B the potential difference between the points is

$$V_{AB} = -\int_{x_A}^{x_B} E_x dx - \int_{y_A}^{y_B} E_y dy - \int_{z_A}^{z_B} E_z dz$$

Each of these integrals is along one coordinate axis and contains only that coordinate variable.

If the change in potential along a small element of the *x*-axis is

$$dV = -E_x dx \rightarrow E_x = -\frac{dV}{dx}$$

This is **- potential gradient** along the *x*-axis

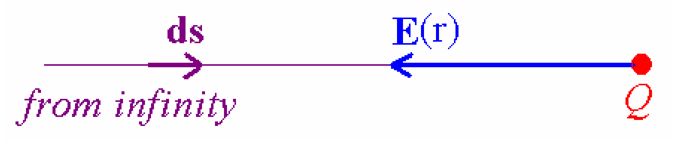
Similarly $$E_y = -\frac{dV}{dy} \text{ and } E_z = -\frac{dV}{dz}$$

Then $$\mathbf{E} = \mathbf{i}E_x + \mathbf{j}E_y + \mathbf{k}E_z = -\left(\mathbf{i}\frac{dV}{dx} + \mathbf{j}\frac{dV}{dy} + \mathbf{k}\frac{dV}{dz}\right)$$

This is **- overall potential gradient**

Benson 25.3

Potential due to a single isolated point charge:



By symmetry the electric field is radial so can form the potential line integral along any radial axis to get potential at r

$$V(r) = -\int_{\infty}^{r} \mathbf{E.ds} = +\int_{\infty}^{r} \frac{Q}{4\pi\varepsilon_0 r^2} dr \quad \text{since } dr = -ds$$

$$= +\frac{Q}{4\pi\varepsilon_0} \int_{\infty}^{r} \frac{dr}{r^2} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r} - \frac{1}{\infty}\right]$$

$$= \frac{Q}{4\pi\varepsilon_0 r}$$

The potentials at two different radii r_1 and r_2 will be

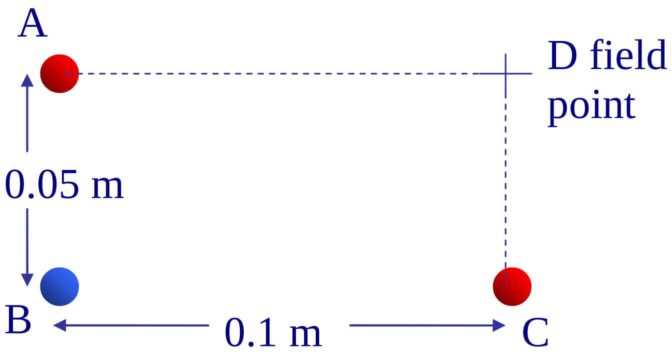
$$V_1 = \frac{Q}{4\pi\varepsilon_0 r_1} \quad \text{and} \quad V_2 = \frac{Q}{4\pi\varepsilon_0 r_2}$$

The potential difference between these two radial positions is

$$V_{12} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_2} - \frac{1}{r_1}\right]$$

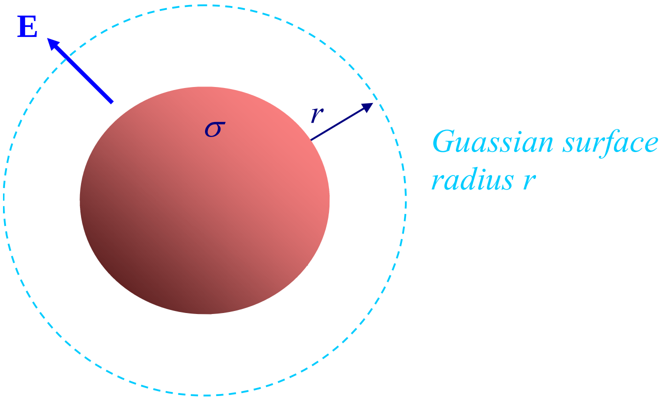
For several individual point charges at different positions by superposition of individual scalar potentials (the scalar sum) the resultant potential is

$$V = \sum_n V_n = \frac{1}{4\pi\varepsilon_0} \sum_n \frac{Q_n}{r_n}$$



Ex: If charge A = 2 μC, charge B = -1 μC, charge C = 1 μC find the potential at field point D. Ans: 189.2 kV

Benson 25.5



Ex: Find the potential outside and inside a sphere of charge radius a and charge density σ.

Ans:

$$\frac{a^3\sigma}{3\varepsilon_0 r} \text{ and } \frac{\sigma}{3\varepsilon_0}\left(\frac{3a^2 - r^2}{2}\right)$$

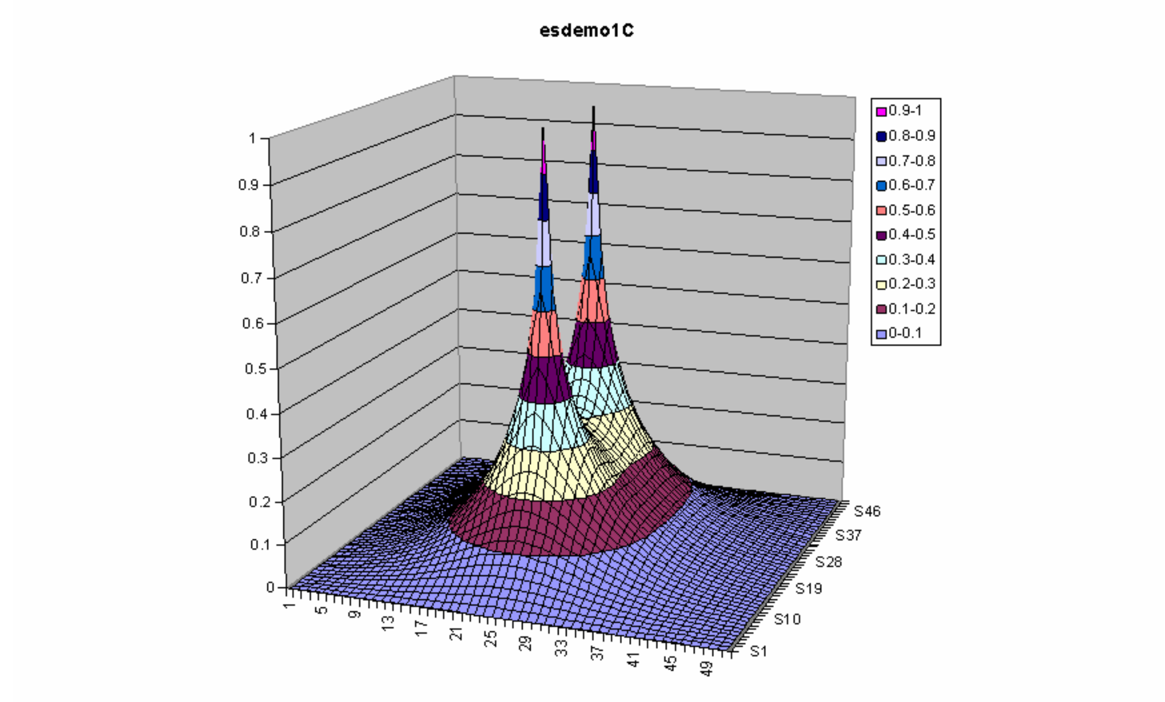
For a spherical surface centred on an isolated point charge the surface is at a constant radius from the charge so that everywhere on the surface is at the same potential.

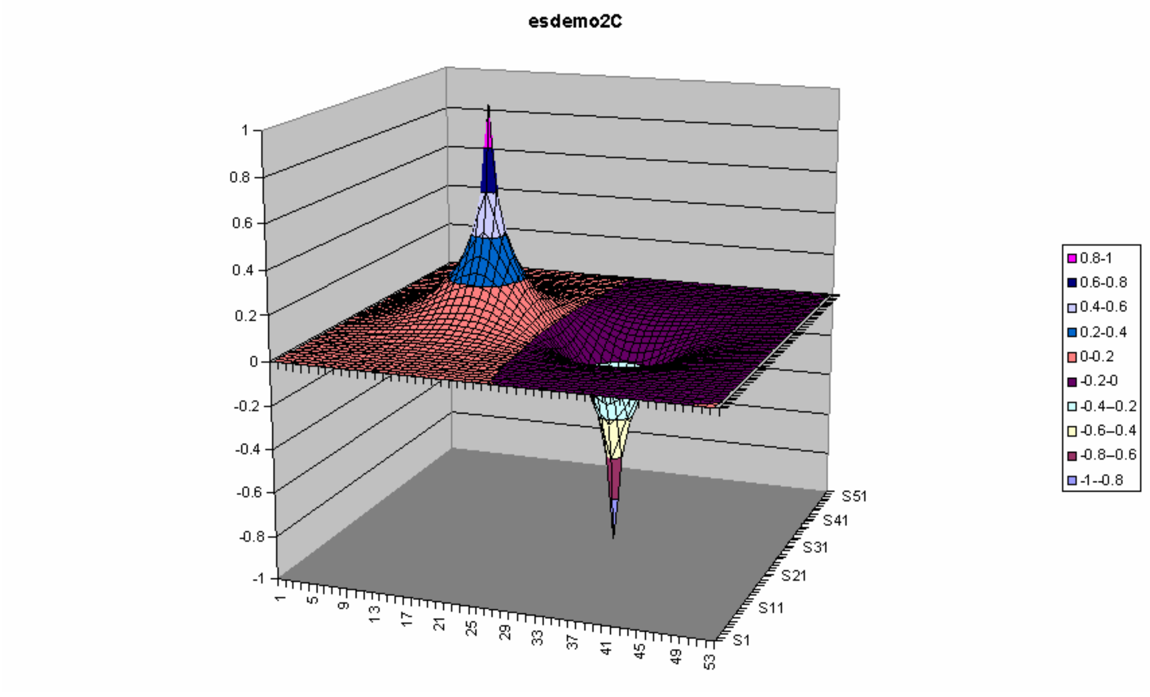
This is an example of an **equipotential surface.**

A general equipotential surface is a 3-dimensional “map” of the surface which has every point at the same chosen potential.

This provides a field “energy mapping” in a similar way to the “force mapping” provided by lines of force.

Ex: Sketch the potential variations due to (a) two equal magnitude same sign point charges, and (b) two equal magnitude opposite sign point charges.





At the surface of a conductor there is no electric field along the conductor surface so there is no potential variation over the surface – the surface of a conductor is an equipotential surface.

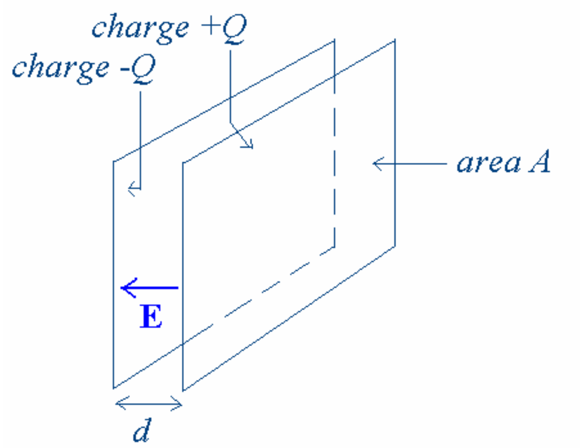
Since there is no electric field inside a conductor all points within the conductor must be at the same potential.

Ex: If the breakdown electric field strength for dry air is 3 MVm^{-1} find the minimum radius and maximum potential for a conducting sphere which has to carry a charge of 1μC.

Ans: 54.7 mm and 164 kV

Potential difference between conductors

Potential difference between two **parallel, flat, conducting plates**:



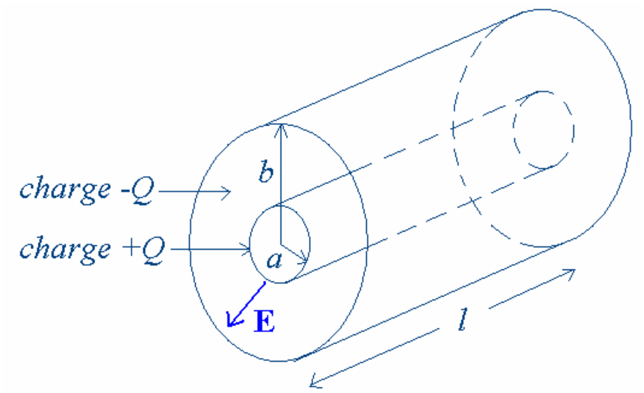
Take a Gaussian surface between the plates and just within one of the plates:

$$\oint_S \mathbf{E.dS} = EA = \frac{Q}{\varepsilon_0} \rightarrow E = \frac{Q}{A\varepsilon_0}$$

The potential difference between the plates is

$$V = -\int_0^d \mathbf{E.ds} = Ed = \frac{d}{\varepsilon_0 A}Q$$

Potential difference between two **coaxial cylindrical conductors**:



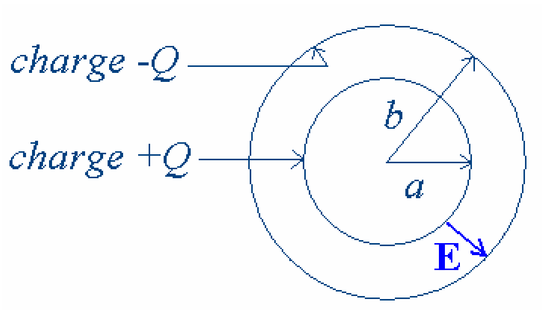
By symmetry the electric field is radial. Take a cylindrical Gaussian surface:

$$\oint_S \mathbf{E.dS} = E2\pi rl = \frac{Q}{\varepsilon_0} \rightarrow E = \frac{Q}{2\pi\varepsilon_0 rl}$$

Potential difference between the conductors is

$$V = \int_a^b Edr = \frac{Q}{2\pi\varepsilon_0 l}\int_a^b \frac{dr}{r} = \frac{Q}{2\pi\varepsilon_0 l}[\ln r]_a^b = \frac{\ln\frac{b}{a}}{2\pi\varepsilon_0 l}Q$$

Potential difference between two **concentric conducting spheres**:



By symmetry electric field is radial. On a spherical Gaussian surface

$$\oint_S \mathbf{E.dS} = E4\pi r^2 = \frac{Q}{\varepsilon_0} \rightarrow E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

The potential difference between the conductors is

$$V = \int_a^b Edr = \frac{Q}{4\pi\varepsilon_0}\int_a^b \frac{dr}{r^2} = \frac{Q}{4\pi\varepsilon_0}\left[-\frac{1}{r}\right]_a^b = \frac{\left(\frac{1}{a}-\frac{1}{b}\right)}{4\pi\varepsilon_0}Q$$

In each of the above examples there is seen to be a linear relationship between the charge applied to the system and the potential difference produced between the conductors.

The particular relationship is determined by a term depending on the form and size of the conductors which always has the dimensions of $\varepsilon_0 \times \text{length}$.

These are particular examples of a general electric field relationship that can be written as

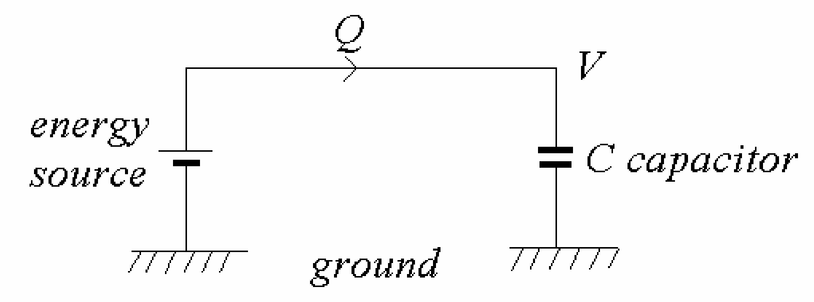
$$\boxed{Q = CV}$$

This defines the **capacitance** of the system C

Capacitance is a property of the conductor system and the material within it.

The unit of capacitance is the **Farad** (F) = Coulombs per volt, C/V

Energy stored in capacitance



It is the convention to take “ground” (= “earth”) as being at zero potential – the earth being considered so large it cannot be “charged”.

One of the conductors of the capacitor that provides the capacitance is connected to ground (0 V), the other conductor is connected to a source of electrical energy represented by a battery.

The energy source provides the energy necessary to move charge onto the conductor at potential V.

Work done to add charge to capacitance is

$$dW = VdQ = CVdV$$

If the capacitance is charged to raise the potential from 0 to V the work done is

$$W = \int_0^V CVdV = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C}$$

Putting charge onto a conductor can be regarded as **doing work to set up its electric field which stores the energy:**

$$C = \frac{\varepsilon_0 A}{d}, \quad V = Ed, \quad W = \frac{1}{2} C V^2 = \frac{1}{2} \varepsilon_0 A d E^2$$

Field **energy density** is then $\boxed{\frac{1}{2} \varepsilon_0 E^2}$

This is a general result for the electric field.

Ex: Find the energy in the field between coaxial cylindrical conductors.

$$C = \frac{2\pi l \varepsilon_0}{\ln\left(\frac{b}{a}\right)}, \quad \text{energy density} = \frac{Q^2}{8\pi^2 l^2 \varepsilon_0} \frac{1}{r^2}, \quad W = \frac{Q^2}{4\pi l \varepsilon_0} \ln\left(\frac{b}{a}\right)$$

If a dielectric (insulating) material is placed in the electric field of a capacitance which is connected to an electrical energy source charge flows from the source to the conductors and the energy stored in the capacitance is increased.

This results from the behaviour of the molecules of the dielectric material in the electric field.

Most molecules have a small electric dipole moment. Some molecules (e.g. water) have a significant dipole moment.

With no electric field molecular orientation is random because of thermal motion.

If an electric field is present there is a tendency for molecular alignment along the field. This stores energy in the dipole-field system thus increasing the energy in the capacitance.

The **relative permittivity** ε_r is the material property describing this effect.

The relative permittivity is the ratio of stored energy with the dielectric material filling the capacitance field to the energy with the field in vacuum.

$$\therefore \varepsilon_r = \frac{W_d}{W_0} = \frac{\frac{1}{2}C_d V^2}{\frac{1}{2}C_0 V^2} = \frac{C_d}{C_0}$$

The capacitance is proportional to the relative permittivity.

Relative permittivities:

Plastics, glass, oils: 2 – 2.5

Polar liquids (water, ammonia): 25 -80

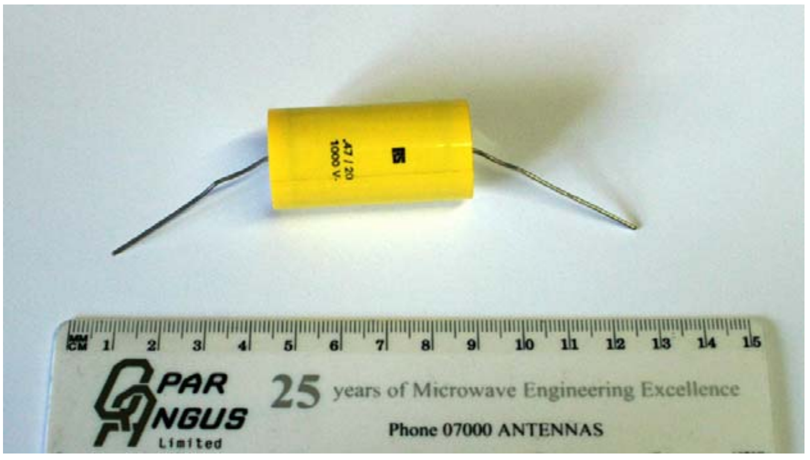
Ceramics (alumina): 10 - 200

Ex: Find the capacitance between two concentric spherical conductors when the space between them is filled with a insulating liquid.

Inner radius = 500 mm, outer radius = 5.5 mm, ε_r = 2

Ans: 5.62 nF

Capacitors are very widely used to perform essential functions in circuits from micro-electronics to international power supplies.



A common form of construction is metallised layers (evaporated aluminium) on a long strip of thin plastic film which is rolled into a cylinder.

Ex: If a metallised plastic film, relative permittivity 2, is 0.01 mm thick and 50 mm wide, what length is needed to provide a capacitance of 1 μF?

Ans: 11.3 m